

# On the continuum and lattice formulations of $N = 4$ $D = 3$ twisted super Yang-Mills

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**ABSTRACT:** Employing a twisted superspace with eight supercharges, we describe an off-shell formulation of  $N = 4$   $D = 3$  twisted super Yang-Mills in the continuum spacetime which underlies the recent proposal of  $N = 4$   $D = 3$  twisted super Yang-Mills on a lattice [6]. By a dimensional reduction from the  $N = 2$   $D = 4$ , we explore the two possible topological twists of  $N = 4$   $D = 3$  and then show that the lattice formulation given in [6] is essentially categorized as the B-type. We also show that, amongst the two inequivalent twists of  $N = 4$   $D = 3$ , only the B-type SYM can be realized on the lattice consistently with the Leibniz rule and the gauge covariance on the lattice.

**KEYWORDS:** Lattice Gauge Field Theories, Superspaces, Extended Supersymmetry, Supersymmetric gauge theory.

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**1. Introduction**

It has been two decades since the twisted supersymmetry (SUSY) was first introduced [1]. Although the notion of twisted SUSY was originally proposed in the context of topological field theories [1–3], it has been recently paid much attention also from the lattice SUSY point of view [4–7]. The main purpose of lattice SUSY is to provide a constructive formulation of supersymmetric models. To this end there have been a wide variety of studies addressing this subject [8–11]. On the other hand, it has long been recognized that there are a couple of obstacles in formulating lattice SUSY, such as the breakdown of Leibniz rule and the existence of fermion doubling on the lattice. The twisted SUSY is providing a fundamental framework to overcome these difficulties. In the series of studies [4–6], we formulated the  $N = D = 2$  twisted Wess-Zumino type models,  $N = D = 2$  twisted super Yang-Mills (SYM) and  $N = 4$   $D = 3$  twisted SYM on a lattice. Starting from a careful observation of difference operators and supercharges, we introduced the notion of lattice Leibniz rule conditions. We then explicitly showed that the Dirac-Kähler twisted SUSY algebra can satisfy these conditions. As a result, we could realize all the supercharges and the invariance on the lattice by systematically introducing the link (anti-)commutator algebra. It is crucial to observe that the importance of the twisted SUSY can be traced back to the intrinsic relation between twisted fermions and Dirac-Kähler fermions [12–14]. Furthermore, the very recent development in terms of the matrix formulation [15] is serving as a fundamental framework realizing the above picture more rigidly. It is also interesting to mention that the above link approach of lattice SUSY and the so-called orbifold approach of lattice SUSY [10] turn to be closely related each other, which was mentioned in the discussion in [5] and also recently pointed out in [16].

Keeping these circumstances in mind, in this paper, we describe a manifestly off-shell formulation of the  $N = 4$   $D = 3$  twisted SYM in the continuum flat spacetime which

underlies our recent proposal of the  $N = 4$   $D = 3$  twisted SYM on the lattice. Studies of the  $N = 4$   $D = 3$  twisted SYM in the continuum spacetime have been given in the past [17, 18] with the classification of two inequivalent topological twists which are called the super BF type (A-type) and the Blau-Thompson type (B-type). The twisted SYM multiplets and the algebra given in these studies are not entirely off-shell. In this paper, we explore these inequivalent twists entirely in a off-shell regime. We then show that the lattice SYM multiplet given in [6] is categorized as the B-type twist. We also investigate these two inequivalent twists from the lattice point of view and show that only the B-type multiplet can be realized on the lattice consistently with the Leibniz rule and gauge covariance on the lattice.

This paper is organized as follows. In section 2, starting from the  $N = 4$   $D = 3$  SUSY algebra, we introduce a twisted  $N = 4$   $D = 3$  superspace formulation. We employ an extended SUSY superconnection method in order to provide a manifest gauge covariant off-shell framework. Introducing the  $N = 4$   $D = 3$  twisted SYM constraints for the supercovariant derivatives, we construct a manifestly invariant formulation of  $N = 4$   $D = 3$  twisted SYM in the continuum spacetime which underlies the recent proposal of twisted SYM on a three dimensional lattice [6]. We also discuss about the twisted SUSY exact relation between the super Chern-Simons and the SYM completely in the off-shell regime, stressing that the existence of sub-algebra and sub-multiplet is responsible for the off-shell super Chern-Simons realization. In section 3, by a dimensional reduction from the  $N = 2$   $D = 4$  twisted SYM, we explore the two possible twists of  $N = 4$   $D = 3$ . We then show that the lattice formulation of SYM given in [6] is classified as the B-type twisted SYM. In section 4, after reviewing the notion of the lattice Leibniz rule [4–6], we examine the possibilities of realizing the twisted  $N = 2$   $D = 4$ ,  $N = 4$   $D = 3$  A-type and B-type SYM on the lattice. We then explicitly show that only the B-type twisted SYM can be consistent with the lattice Leibniz rule and the gauge covariance on the lattice. Namely, the formulation given in [6] is the unique lattice realization of  $N = 4$   $D = 3$  twisted SYM satisfying these conditions. Section 5 gives the summary and the discussions.

## 2. Superspace formulation of $N = 4$ $D = 3$ twisted SYM

In this section, employing a twisted superfield method with eight supercharges, we explicitly perform an off-shell construction of the  $N = 4$   $D = 3$  twisted SYM. It provides the underlying continuum theory for the recent proposal of twisted SYM on the three dimensional lattice [6]. We start from the following  $N = 4$   $D = 3$  SUSY algebra,

$$\{Q_{\alpha i}, \bar{Q}_{j\beta}\} = 2\delta_{ij}(\gamma_{\mu})_{\alpha\beta}P_{\mu}, \tag{2.1}$$

$$[J_{\mu}, Q_{\alpha i}] = +\frac{1}{2}(\gamma_{\mu})_{\alpha\beta}Q_{\beta i}, \quad [J_{\mu}, \bar{Q}_{i\alpha}] = -\frac{1}{2}\bar{Q}_{i\beta}(\gamma_{\mu})_{\beta\alpha}, \tag{2.2}$$

$$[R_{\mu}, Q_{\alpha i}] = -\frac{1}{2}Q_{\alpha j}(\gamma_{\mu})_{ji}, \quad [R_{\mu}, \bar{Q}_{i\alpha}] = +\frac{1}{2}(\gamma_{\mu})_{ij}\bar{Q}_{j\alpha}, \tag{2.3}$$

$$[J_{\mu}, P_{\nu}] = -i\epsilon_{\mu\nu\rho}P_{\rho}, \quad [J_{\mu}, J_{\nu}] = -i\epsilon_{\mu\nu\rho}J_{\rho}, \quad [R_{\mu}, R_{\nu}] = -i\epsilon_{\mu\nu\rho}R_{\rho}, \tag{2.4}$$

$$[R_{\mu}, P_{\nu}] = [P_{\mu}, P_{\nu}] = [J_{\mu}, R_{\nu}] = 0, \tag{2.5}$$

where the gamma matrices  $\gamma_\mu$  can be taken as the Pauli matrices,  $\gamma^\mu (\mu = 1, 2, 3) \equiv (\sigma^1, \sigma^2, \sigma^3)$ . The conjugate supercharge  $\bar{Q}_{i\alpha}$  can be taken as the complex conjugation of  $Q_{\alpha i}$ ,  $\bar{Q}_{i\alpha} = Q_{\alpha i}^*$ . The  $J_\mu$  and  $R_\mu$  ( $\mu = 1, 2, 3$ ) are the generators of  $\text{SO}(3)_E \simeq \text{SU}(2)_E$  Euclidean Lorentz rotations and  $\text{SO}(3)_R \simeq \text{SU}(2)_R$  internal rotations, respectively.

As in the case of  $N = D = 2$  [12, 13, 19] and  $N = D = 4$  [14, 17, 20], the twisting procedure can be performed by taking the diagonal subgroup of the Lorentz rotations and the internal rotations. Here in the case of  $N = 4$   $D = 3$ , we take the diagonal subgroup  $(\text{SO}(3)_E \times \text{SO}(3)_R)_{\text{diag}}$  whose covering group is  $(\text{SU}(2)_E \times \text{SU}(2)_R)_{\text{diag}}$ . This corresponds to introducing the twisted Lorentz generator  $J_\mu^{\text{diag}}$  as a diagonal sum of  $J_\mu$  and  $R_\mu$ ,  $J_\mu^{\text{diag}} \equiv J_\mu + R_\mu$ . Since, after the twisting, the Lorentz index  $\alpha$  and internal index  $i$  are rotated on the same footing, the resulting algebra is most naturally expressed in terms of the following Dirac-Kähler expansion of the supercharges,

$$Q_{\alpha i} = (\mathbf{1}s + \gamma_\mu s_\mu)_{\alpha i}, \quad \bar{Q}_{i\alpha} = (\mathbf{1}\bar{s} + \gamma_\mu \bar{s}_\mu)_{i\alpha}, \quad (2.6)$$

where  $\mathbf{1}$  represents the two-by-two unit matrix. The coefficients  $(s, \bar{s}_\mu, s_\mu, \bar{s})$  are called the  $N = 4$   $D = 3$  twisted supercharges. After the expansions, the original SUSY algebra (2.1) can be expressed as,

$$\{s, \bar{s}_\mu\} = P_\mu, \quad (2.7)$$

$$\{s_\mu, \bar{s}_\nu\} = -i\epsilon_{\mu\nu\rho} P_\rho, \quad (2.8)$$

$$\{\bar{s}, s_\mu\} = P_\mu, \quad (2.9)$$

$$\{\text{others}\} = 0, \quad (2.10)$$

where  $\epsilon_{\mu\nu\rho}$  is the three dimensional totally anti-symmetric tensor with  $\epsilon_{123} = +1$ . The Lorentz and the internal rotations of the supercharges are re-expressed on the twisted basis,

$$[J_\mu, s] = +\frac{1}{2}s_\mu, \quad [J_\mu, s_\nu] = -\frac{i}{2}\epsilon_{\mu\nu\rho}s_\rho + \frac{1}{2}\delta_{\mu\nu}s, \quad (2.11)$$

$$[J_\mu, \bar{s}] = -\frac{1}{2}\bar{s}_\mu, \quad [J_\mu, \bar{s}_\nu] = -\frac{i}{2}\epsilon_{\mu\nu\rho}\bar{s}_\rho - \frac{1}{2}\delta_{\mu\nu}\bar{s}, \quad (2.12)$$

$$[R_\mu, s] = -\frac{1}{2}s_\mu, \quad [R_\mu, s_\nu] = -\frac{i}{2}\epsilon_{\mu\nu\rho}s_\rho - \frac{1}{2}\delta_{\mu\nu}s, \quad (2.13)$$

$$[R_\mu, \bar{s}] = +\frac{1}{2}\bar{s}_\mu, \quad [R_\mu, \bar{s}_\nu] = -\frac{i}{2}\epsilon_{\mu\nu\rho}\bar{s}_\rho + \frac{1}{2}\delta_{\mu\nu}\bar{s}. \quad (2.14)$$

Notice that  $(s, \bar{s})$  and  $(s_\mu, \bar{s}_\mu)$  transform as scalars and vectors under  $(\text{SO}(3)_E \times \text{SO}(3)_R)_{\text{diag}}$ , respectively. Namely, under the twisted Lorentz generator  $J_\mu^{\text{diag}} = J_\mu + R_\mu$  they transform as

$$[J_\mu^{\text{diag}}, s] = [J_\mu^{\text{diag}}, \bar{s}] = 0, \quad [J_\mu^{\text{diag}}, s_\nu] = -i\epsilon_{\mu\nu\rho}s_\rho, \quad [J_\mu^{\text{diag}}, \bar{s}_\nu] = -i\epsilon_{\mu\nu\rho}\bar{s}_\rho. \quad (2.15)$$

Once we have the SUSY algebra of  $(s, \bar{s}_\mu, s_\mu, \bar{s})$ , we can construct the corresponding superspace by introducing the fermionic coordinates  $\theta_A = (\theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta})$ . The  $N = D = 2$  and  $N = D = 4$  Dirac-Kähler twisted superspace formulations are elaborated in [13, 14]. Here we begin by considering the following supergroup element of the twisted  $N = 4$   $D = 3$ ,

$$G(x_\mu, \theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta}) = e^{i(-x_\mu P_\mu + \theta s + \bar{\theta}_\mu \bar{s}_\mu + \theta_\mu s_\mu + \bar{\theta} \bar{s})}. \quad (2.16)$$

By using the algebra (2.7)–(2.10), we have,

$$G(0, \xi, \bar{\xi}_\mu, \xi_\mu, \bar{\xi})G(x_\mu, \theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta}) = G(x_\mu + a_\mu, \theta + \xi, \bar{\theta}_\mu + \bar{\xi}_\mu, \theta_\mu + \xi_\mu, \bar{\theta} + \bar{\xi}), \quad (2.17)$$

where the variations of the bosonic coordinates,  $a_\mu$ , are given in terms of the combinations of the fermionic coordinates,

$$a_\rho = \frac{i}{2}\xi\bar{\theta}_\rho + \frac{i}{2}\bar{\xi}\theta_\rho + \frac{i}{2}\xi_\rho\bar{\theta} + \frac{i}{2}\bar{\xi}_\rho\theta + \frac{1}{2}\epsilon_{\mu\nu\rho}\xi_\mu\bar{\theta}_\nu - \frac{1}{2}\epsilon_{\mu\nu\rho}\bar{\xi}_\mu\theta_\nu. \quad (2.18)$$

Expanding the r.h.s. of (2.17) w.r.t.  $(\xi, \bar{\xi}_\mu, \xi_\mu, \bar{\xi})$ ,

$$G(x_\mu + a_\mu, \theta + \xi, \bar{\theta}_\mu + \bar{\xi}_\mu, \theta_\mu + \xi_\mu, \bar{\theta} + \bar{\xi}) = (\xi Q + \bar{\xi}_\mu \bar{Q}_\mu + \xi_\mu Q_\mu + \bar{\xi} \bar{Q})G(x_\mu, \theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta}), \quad (2.19)$$

one finds the superspace expressions of the  $N = 4$   $D = 3$  twisted SUSY generators  $(Q, \bar{Q}_\mu, Q_\mu, \bar{Q})$ ,

$$Q = \frac{\partial}{\partial\theta} + \frac{i}{2}\bar{\theta}_\mu\partial_\mu, \quad Q_\mu = \frac{\partial}{\partial\theta_\mu} + \frac{i}{2}\bar{\theta}\partial_\mu + \frac{1}{2}\epsilon_{\mu\nu\rho}\bar{\theta}_\nu\partial_\rho, \quad (2.20)$$

$$\bar{Q} = \frac{\partial}{\partial\bar{\theta}} + \frac{i}{2}\theta_\mu\partial_\mu, \quad \bar{Q}_\mu = \frac{\partial}{\partial\bar{\theta}_\mu} + \frac{i}{2}\theta\partial_\mu - \frac{1}{2}\epsilon_{\mu\nu\rho}\theta_\nu\partial_\rho, \quad (2.21)$$

which satisfy the following algebra,

$$\{Q, \bar{Q}_\mu\} = +i\partial_\mu, \quad (2.22)$$

$$\{Q_\mu, \bar{Q}_\nu\} = +\epsilon_{\mu\nu\rho}\partial_\rho, \quad (2.23)$$

$$\{\bar{Q}, Q_\mu\} = +i\partial_\mu, \quad (2.24)$$

$$\{\text{others}\} = 0. \quad (2.25)$$

Note that the above SUSY generators are induced by the left multiplication of the supergroup element (2.17). In contrast, we also have the following  $N = 4$   $D = 3$  superderivatives  $(D, \bar{D}_\mu, D_\mu, \bar{D})$  which are induced by the right multiplication of the supergroup element,

$$G(x_\mu, \theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta})G(0, \xi, \bar{\xi}_\mu, \xi_\mu, \bar{\xi}) = (\xi D + \bar{\xi}_\mu \bar{D}_\mu + \xi_\mu D_\mu + \bar{\xi} \bar{D})G(x_\mu, \theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta}), \quad (2.26)$$

$$D = \frac{\partial}{\partial\theta} - \frac{i}{2}\bar{\theta}_\mu\partial_\mu, \quad D_\mu = \frac{\partial}{\partial\theta_\mu} - \frac{i}{2}\bar{\theta}\partial_\mu - \frac{1}{2}\epsilon_{\mu\nu\rho}\bar{\theta}_\nu\partial_\rho, \quad (2.27)$$

$$\bar{D} = \frac{\partial}{\partial\bar{\theta}} - \frac{i}{2}\theta_\mu\partial_\mu, \quad \bar{D}_\mu = \frac{\partial}{\partial\bar{\theta}_\mu} - \frac{i}{2}\theta\partial_\mu + \frac{1}{2}\epsilon_{\mu\nu\rho}\theta_\nu\partial_\rho. \quad (2.28)$$

The superderivatives satisfy the following algebra,

$$\{D, \bar{D}_\mu\} = -i\partial_\mu, \quad (2.29)$$

$$\{D_\mu, \bar{D}_\nu\} = -\epsilon_{\mu\nu\rho}\partial_\rho, \quad (2.30)$$

$$\{\bar{D}, D_\mu\} = -i\partial_\mu, \quad (2.31)$$

$$\{\text{others}\} = 0. \quad (2.32)$$

It is important to note that the SUSY generators  $Q_A = (Q, \bar{Q}_\mu, Q_\mu, \bar{Q})$  and the superderivatives  $D_A = (D, \bar{D}_\mu, D_\mu, \bar{D})$  anti-commute each other,

$$\{Q_A, D_B\} = 0. \tag{2.33}$$

Having these superspace operators in hand, we then proceed to formulate the  $N = 4$   $D = 3$  twisted SYM by means of the superfield method. One of the most systematic treatments to construct the supersymmetric gauge theories is the so-called super-connection method which was introduced in [22]. It was also applied to the investigation of topological quantum field theory [23]. The detailed analysis of super-connection formulations for twisted  $N = D = 2$  and  $N = 2$   $D = 4$  from the Dirac-Kähler point of view are given in [14, 21].

We first introduce the  $N = 4$   $D = 3$  fermionic gauge covariant derivatives  $\nabla_A = (\nabla, \bar{\nabla}_\mu, \nabla_\mu, \bar{\nabla})$ ,

$$\nabla = D - i\Gamma(x, \theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta}), \quad \nabla_\mu = D_\mu - i\Gamma_\mu(x, \theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta}), \tag{2.34}$$

$$\bar{\nabla} = \bar{D} - i\bar{\Gamma}(x, \theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta}), \quad \bar{\nabla}_\mu = \bar{D}_\mu - i\bar{\Gamma}_\mu(x, \theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta}), \tag{2.35}$$

where  $\Gamma_A = (\Gamma, \bar{\Gamma}_\mu, \Gamma_\mu, \bar{\Gamma})$  are denoting the superconnections associated with the superderivatives. All of the  $\Gamma_A$ 's are the functions of  $(x, \theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta})$  and are belonging to the adjoint representation of the gauge group. The fermionic gauge covariant derivatives  $\nabla_A$  are transforming under the supergauge transformations as follows,

$$\nabla_A \rightarrow \nabla'_A = e^{-i\Omega} \nabla_A e^{+i\Omega}, \tag{2.36}$$

where  $\Omega = \Omega(x, \theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta})$  denotes the generic hermitian superfield as we will see later on. Since the above super-connections  $\Gamma_A$  contain a large number of component fields,

$$\Gamma_A = \psi_A + \theta_B \psi_{BA} + \theta_C \theta_B \psi_{CBA} + \dots, \tag{2.37}$$

the resulting multiplet might become highly reducible in general even after taking the Wess-Zumino gauge. The central issue of formulating the extended supersymmetric gauge theories is thus how to reduce the number of component fields and how to obtain the irreducible SUSY multiplet in a gauge covariant manner.

One of the possible ways to obtain such an irreducible  $N = 4$   $D = 3$  twisted SYM multiplet is to impose the following constraints on the fermionic gauge covariant derivatives,

$$\{\nabla, \bar{\nabla}_\mu\} = -i(\nabla_{\underline{\mu}} - i\Phi^{(\mu)}), \tag{2.38}$$

$$\{\nabla_\mu, \bar{\nabla}_\nu\} = -\epsilon_{\mu\nu\rho}(\nabla_{\underline{\rho}} + i\Phi^{(\rho)}), \tag{2.39}$$

$$\{\bar{\nabla}, \nabla_\mu\} = -i(\nabla_{\underline{\mu}} - i\Phi^{(\mu)}), \tag{2.40}$$

$$\{others\} = 0. \tag{2.41}$$

The  $\nabla_{\underline{\mu}}$  and  $\Phi^{(\mu)}$  are the superfields whose lowest components are representing the gauge covariant derivatives and the scalar fields, respectively,

$$\nabla_{\underline{\mu}} = \partial_\mu - iA_\mu + \dots, \tag{2.42}$$

$$\Phi^{(\mu)} = \phi^{(\mu)} + \dots, \tag{2.43}$$

here and in the following the dots  $\dots$  are representing the possible higher order terms w.r.t. the fermionic coordinates  $\theta_A = (\theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta})$ . All of the components in the superfields  $\nabla_{\underline{\mu}}$  and  $\Phi^{(\mu)}$ , including  $A_\mu$  and  $\phi^{(\mu)}$ , can be essentially expressed in terms of the component fields embedded in the superconnections  $\Gamma_A = (\Gamma, \bar{\Gamma}_\mu, \Gamma_\mu, \bar{\Gamma})$ .

There are several remarks in order. First, since in the constraints (2.38)–(2.41) we introduced the scalar fields  $\phi^{(\mu)}$  on the same footing as the gauge fields  $A_\mu$ , one may wonder how these fields are transformed under the Lorentz  $\text{SO}(3)_E$  and the internal  $\text{SO}(3)_R$  rotations. Reminding that the fermionic covariant derivatives  $\nabla_A = (\nabla, \bar{\nabla}_\mu, \nabla_\mu, \bar{\nabla})$  are transforming just like as  $s_A = (s, \bar{s}_\mu, s_\mu, \bar{s})$  in (2.11)–(2.14), respectively, one finds from (2.38)–(2.41),

$$[J_\mu, \nabla_{\underline{\nu}} \pm i\Phi^{(\nu)}] = -\frac{i}{2}\epsilon_{\mu\nu\rho} (\nabla_{\underline{\rho}} + i\Phi^{(\rho)}) - \frac{i}{2}\epsilon_{\mu\nu\rho} (\nabla_{\underline{\rho}} - i\Phi^{(\rho)}) \quad (2.44)$$

$$= -i\epsilon_{\mu\nu\rho} \nabla_{\underline{\rho}}, \quad (2.45)$$

from which it obeys,

$$[J_\mu, \nabla_{\underline{\nu}}] = -i\epsilon_{\mu\nu\rho} \nabla_{\underline{\rho}}, \quad [J_\mu, \Phi^{(\nu)}] = 0. \quad (2.46)$$

Namely, the gauge fields  $A_\mu$  are actually transforming as a  $\text{SO}(3)_E$  vector while the scalar fields  $\phi^{(\mu)}$  are transforming as  $\text{SO}(3)_E$  scalars. In contrast, one could also obtain

$$[R_\mu, \nabla_{\underline{\nu}}] = 0, \quad [R_\mu, \Phi^{(\nu)}] = -i\epsilon_{\mu\nu\rho} \Phi^{(\rho)}, \quad (2.47)$$

which implies that the  $A_\mu$  and  $\phi^{(\mu)}$  are transforming as  $\text{SO}(3)_R$  scalars and a vector, respectively. One can thus see that even though the  $\phi^{(\mu)}$  are introduced on the same footing as the gauge fields  $A_\mu$ , they are appropriately transforming as scalars of the original Lorentz rotations. Obviously, the sign difference in front of the  $\Phi^{(\mu)}$  in (2.38)–(2.40) is responsible for these transformation properties. Furthermore, one should notice that, after the twisting, both of  $A_\mu$  and  $\phi^{(\mu)}$  transform as vectors under the twisted rotational group  $(\text{SO}(3)_E \times \text{SO}(3)_R)_{\text{diag}}$ ,

$$[J_\mu^{\text{diag}}, \nabla_{\underline{\nu}}] = -i\epsilon_{\mu\nu\rho} \nabla_{\underline{\rho}}, \quad [J_\mu^{\text{diag}}, \Phi^{(\nu)}] = -i\epsilon_{\mu\nu\rho} \Phi^{(\rho)}. \quad (2.48)$$

The combinations  $\nabla_{\underline{\mu}} \mp i\Phi^{(\mu)}$  appeared in the r.h.s. of (2.38)–(2.40) are thus the covariant expressions w.r.t. the twisted rotational group  $(\text{SO}(3)_E \times \text{SO}(3)_R)_{\text{diag}}$ .

The second remark is regarding the hermiticity of the constraints (2.38)–(2.41). We impose the following hermitian conjugation properties on the fermionic covariant derivatives,

$$\nabla^\dagger = \bar{\nabla}, \quad \nabla_\mu^\dagger = \bar{\nabla}_\mu, \quad (2.49)$$

which are consistent with the complex conjugation nature of the supercharges  $Q_{\alpha i}$  and  $\bar{Q}_{i\alpha}$  in (2.1),  $Q_{\alpha i}^* = \bar{Q}_{i\alpha}$ . One could easily notice that, in order to be compatible with the constraints (2.38)–(2.40), the supergauge transformation  $\Omega$  in (2.36) should be hermitian.

Correspondingly, one can take the  $A_\mu$  and the scalars  $\phi^{(\mu)}$  as the hermitian fields which transform under the gauge transformation as,

$$\partial_\mu - iA_\mu \rightarrow e^{-i\omega}(\partial_\mu - iA_\mu)e^{+i\omega}, \quad \phi^{(\mu)} \rightarrow e^{-i\omega}\phi^{(\mu)}e^{+i\omega}, \quad (2.50)$$

where  $\omega$  denotes the  $\theta_A$  independent first component of the supergauge transformation  $\Omega$  satisfying  $\omega^\dagger = \omega$ .

The third remark is that the constraints (2.38)–(2.41) corresponds to the naive continuum limit of the  $N = 4$   $D = 3$  lattice SYM constraints recently proposed in [6], where the gauge fields  $A_\mu$  are exponentiated together with the scalar fields  $\phi^{(\mu)}$  such that they could represent the bosonic gauge link variables either of the forward or the backward type,

$$\partial_\mu - i \left( A_\mu \pm \phi^{(\mu)} \right) \rightarrow \mp \left( e^{\pm i(A_\mu \pm \phi^{(\mu)})} \right)_{x \pm n_\mu, x}, \quad (2.51)$$

where the subscripts indicate that they are located on links from  $x$  to  $x + n_\mu$  (forward) and  $x$  to  $x - n_\mu$  (backward), respectively, for a generic site  $x$ . As is also stressed in [4–6], the twisting is playing a fundamental role in realizing the supersymmetry on the lattice and it can be traced back to the intrinsic relation between the twisted fermions and the Dirac-Kähler fermions [12–14]. Here we find the importance of the twisting in the bosonic sector as well. Namely, the exponential forms in (2.51) can transform covariantly only under the twisted rotational group  $(\text{SO}(3)_E \times \text{SO}(3)_R)_{\text{diag}}$  and not under the  $\text{SO}(3)_E$  and  $\text{SO}(3)_R$  independently. We will come back to this point once again in section 4.

Once we impose the constraints (2.38)–(2.41), the whole information of the resulting  $N = 4$   $D = 3$  twisted SYM multiplet can be obtained by analyzing the Jacobi identities together with the constraints (2.38)–(2.41). For the notational simplicity, we re-write the constraints (2.38)–(2.41) as

$$\{\nabla, \bar{\nabla}_\mu\} = -i\nabla_{+\mu}, \quad (2.52)$$

$$\{\nabla_\mu, \bar{\nabla}_\nu\} = -\epsilon_{\mu\nu\rho}\nabla_{-\rho}, \quad (2.53)$$

$$\{\bar{\nabla}, \nabla_\mu\} = -i\nabla_{+\mu}, \quad (2.54)$$

$$\{\text{others}\} = 0. \quad (2.55)$$

The symbols  $\nabla_{\pm\mu}$  are defined by

$$\nabla_{\pm\mu} = \nabla_\mu \mp i\Phi^{(\mu)} \quad (2.56)$$

$$= \partial_\mu - i \left( A_\mu \pm \phi^{(\mu)} \right) + \dots, \quad (2.57)$$

$$= \mathcal{D}_{\pm\mu} + \dots, \quad (2.58)$$

where we denote the  $\theta_A$  independent part of  $\nabla_{\pm\mu}$  as  $\mathcal{D}_{\pm\mu} = \partial_\mu - i(A_\mu \pm \phi^{(\mu)})$ . Since the  $N = 4$   $D = 3$  SYM constraints (2.52)–(2.55) are formally similar to the lattice SYM constraints in [6], the Jacobi identity analysis also goes parallel to the lattice analysis.<sup>1</sup>

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<sup>1</sup>Since in the lattice formulations of SYM [5, 6] all the operators are generically defined on links, all the (anti-)commutators are replaced by the “link” (anti-)commutators. See also the section 4 of this paper.



	$\nabla$	$\bar{\nabla}_1$	$\bar{\nabla}_2$	$\bar{\nabla}_3$	$\nabla_1$	$\nabla_2$	$\nabla_3$	$\bar{\nabla}$
$\nabla_{+1}$	0	0	$+i\Lambda_3$	$-i\Lambda_2$	0	$-i\bar{\Lambda}_3$	$+i\bar{\Lambda}_2$	0
$\nabla_{-1}$	$+\bar{\Lambda}_1$	$-\Upsilon$	0	0	$-\bar{\Upsilon}$	0	0	$+\Lambda_1$
$\nabla_{+2}$	0	$-i\Lambda_3$	0	$+i\Lambda_1$	$+i\bar{\Lambda}_3$	0	$-i\bar{\Lambda}_1$	0
$\nabla_{-2}$	$+\bar{\Lambda}_2$	0	$-\Upsilon$	0	0	$-\bar{\Upsilon}$	0	$+\Lambda_2$
$\nabla_{+3}$	0	$+i\Lambda_2$	$-i\Lambda_1$	0	$-i\bar{\Lambda}_2$	$+i\bar{\Lambda}_1$	0	0
$\nabla_{-3}$	$+\bar{\Lambda}_3$	0	0	$-\Upsilon$	0	0	$-\bar{\Upsilon}$	$+\bar{\Lambda}_3$

**Table 1:** All components of the commutators  $[\nabla_A, \nabla_{\pm\mu}]$  with  $\nabla_A = (\nabla, \bar{\nabla}_\mu, \nabla_\mu, \bar{\nabla})$

The Jacobi identities of three fermionic covariant derivatives give,

$$[\nabla_\mu, \nabla_{+\nu}] + [\nabla_\nu, \nabla_{+\mu}] = 0, \quad [\bar{\nabla}_\mu, \nabla_{+\nu}] + [\bar{\nabla}_\nu, \nabla_{+\mu}] = 0, \quad (2.59)$$

$$[\nabla_\mu, \nabla_{+\nu}] - i\epsilon_{\mu\nu\rho}[\nabla, \nabla_{-\rho}] = 0, \quad [\bar{\nabla}_\mu, \nabla_{+\nu}] + i\epsilon_{\mu\nu\rho}[\bar{\nabla}, \nabla_{-\rho}] = 0, \quad (2.60)$$

$$\epsilon_{\mu\nu\lambda}[\nabla_\rho, \nabla_{-\lambda}] + \epsilon_{\rho\nu\lambda}[\nabla_\mu, \nabla_{-\lambda}] = 0, \quad \epsilon_{\mu\nu\lambda}[\bar{\nabla}_\rho, \nabla_{-\lambda}] + \epsilon_{\rho\nu\lambda}[\bar{\nabla}_\mu, \nabla_{-\lambda}] = 0, \quad (2.61)$$

$$[\nabla, \nabla_{+\mu}] = [\bar{\nabla}, \nabla_{+\mu}] = 0, \quad (2.62)$$

from which we can define the following non-vanishing fermionic superfields  $(\Upsilon, \bar{\Lambda}_\mu, \Lambda_\mu, \bar{\Upsilon})$ ,<sup>2</sup>

$$[\nabla, \nabla_{-\rho}] \equiv +(\bar{\Lambda}_\rho), \quad [\bar{\nabla}, \nabla_{-\rho}] \equiv +(\Lambda_\rho), \quad (2.63)$$

$$[\nabla_\mu, \nabla_{+\nu}] = +i\epsilon_{\mu\nu\rho}(\bar{\Lambda}_\rho), \quad [\bar{\nabla}_\mu, \nabla_{+\nu}] = -i\epsilon_{\mu\nu\rho}(\Lambda_\rho), \quad (2.64)$$

$$[\nabla_\mu, \nabla_{-\nu}] \equiv -\delta_{\mu\nu}(\bar{\Upsilon}), \quad [\bar{\nabla}_\mu, \nabla_{-\nu}] \equiv -\delta_{\mu\nu}(\Upsilon). \quad (2.65)$$

We denote the lowest components of the fermionic superfields  $(\Upsilon, \bar{\Lambda}_\mu, \Lambda_\mu, \bar{\Upsilon})$  as  $(\rho, \bar{\lambda}_\mu, \lambda_\mu, \bar{\rho})$ ,

$$\Upsilon = \rho + \dots, \quad \bar{\Upsilon} = \bar{\rho} + \dots, \quad \Lambda_\mu = \lambda_\mu + \dots, \quad \bar{\Lambda}_\mu = \bar{\lambda}_\mu + \dots. \quad (2.66)$$

The  $(\rho, \bar{\lambda}_\mu, \lambda_\mu, \bar{\rho})$  are representing the  $N = 4$   $D = 3$  twisted fermions in the SYM multiplet. The vanishing conditions resulting from the relations (2.59)–(2.62) also give rise to the covariant “chiral” or “anti-chiral” conditions for  $\nabla_{\pm\mu}$ , for example,

$$[\nabla, \nabla_{+3}] = [\bar{\nabla}_3, \nabla_{+3}] = [\nabla_3, \nabla_{+3}] = [\bar{\nabla}, \nabla_{+3}] = 0, \quad (2.67)$$

$$[\nabla_1, \nabla_{-3}] = [\nabla_2, \nabla_{-3}] = [\bar{\nabla}_1, \nabla_{-3}] = [\bar{\nabla}_2, \nabla_{-3}] = 0. \quad (2.68)$$

One also has the similar conditions for  $\nabla_{\pm 1}$  and  $\nabla_{\pm 2}$ . All the commutators of  $\nabla_A = (\nabla, \bar{\nabla}_\mu, \nabla_\mu, \bar{\nabla})$  and  $\nabla_{\pm\mu}$  are summarized in table 1. As we will see, these conditions are playing important roles when constructing the twisted SUSY invariant action.

<sup>2</sup>For the later convenience, we took the sign conventions of  $(\rho, \bar{\rho})$  oppositely from the ones given in [6].

By taking the anti-commutators of  $\nabla_A$ 's with the relations (2.63)–(2.65), we have,

$$\{\nabla, \Lambda_\mu\} - \frac{1}{2}\epsilon_{\mu\rho\sigma}[\nabla_{+\rho}, \nabla_{+\sigma}] = 0, \quad \{\bar{\nabla}, \bar{\Lambda}_\mu\} + \frac{1}{2}\epsilon_{\mu\rho\sigma}[\nabla_{+\rho}, \nabla_{+\sigma}] = 0, \quad (2.69)$$

$$\{\bar{\nabla}_\mu, \Lambda_\nu\} - \delta_{\mu\nu}\{\bar{\nabla}, \Upsilon\} = 0, \quad \{\nabla_\mu, \bar{\Lambda}_\nu\} - \delta_{\mu\nu}\{\nabla, \bar{\Upsilon}\} = 0, \quad (2.70)$$

$$\epsilon_{\lambda\nu\rho}\{\bar{\nabla}_\mu, \bar{\Lambda}_\rho\} + i\epsilon_{\lambda\mu\rho}[\nabla_{+\nu}, \nabla_{-\rho}] - \epsilon_{\mu\nu\rho}\{\nabla_\lambda, \Lambda_\rho\} = 0, \quad (2.71)$$

$$\delta_{\mu\nu}\{\bar{\nabla}, \bar{\Upsilon}\} + i[\nabla_{-\nu}, \nabla_{+\mu}] - \{\nabla_\mu, \Lambda_\nu\} = 0, \quad (2.72)$$

$$\delta_{\mu\nu}\{\nabla, \Upsilon\} + i[\nabla_{-\nu}, \nabla_{+\mu}] - \{\bar{\nabla}_\mu, \bar{\Lambda}_\nu\} = 0, \quad (2.73)$$

$$\delta_{\lambda\nu}\{\bar{\nabla}_\mu, \bar{\Upsilon}\} + \epsilon_{\lambda\mu\rho}[\nabla_{-\nu}, \nabla_{-\rho}] + \delta_{\mu\nu}\{\nabla_\lambda, \Upsilon\} = 0, \quad (2.74)$$

$$\{\bar{\nabla}, \Lambda_\mu\} = \{\nabla, \bar{\Lambda}_\mu\} = \{\bar{\nabla}_\mu, \Upsilon\} = \{\nabla_\mu, \bar{\Upsilon}\} = 0, \quad (2.75)$$

which can be solved w.r.t. the anti-commutators of the fermionic derivatives  $\nabla_A$  and the fermionic superfields  $(\Lambda_\mu, \bar{\Lambda}_\mu)$ ,

$$\{\nabla, \Lambda_\mu\} = +\frac{1}{2}\epsilon_{\mu\rho\sigma}[\nabla_{+\rho}, \nabla_{+\sigma}], \quad \{\bar{\nabla}, \bar{\Lambda}_\mu\} = -\frac{1}{2}\epsilon_{\mu\rho\sigma}[\nabla_{+\rho}, \nabla_{+\sigma}], \quad (2.76)$$

$$\{\nabla_\mu, \bar{\Lambda}_\nu\} = -\delta_{\mu\nu}\mathcal{G}, \quad \{\bar{\nabla}_\mu, \Lambda_\nu\} = -\delta_{\mu\nu}\bar{\mathcal{G}}, \quad (2.77)$$

$$\{\nabla_\mu, \Lambda_\nu\} = -i[\nabla_{+\mu}, \nabla_{-\nu}] - \delta_{\mu\nu}(\mathcal{K} - \frac{i}{2}[\nabla_{+\rho}, \nabla_{-\rho}]), \quad (2.78)$$

$$\{\bar{\nabla}_\mu, \bar{\Lambda}_\nu\} = -i[\nabla_{+\mu}, \nabla_{-\nu}] + \delta_{\mu\nu}(\mathcal{K} + \frac{i}{2}[\nabla_{+\rho}, \nabla_{-\rho}]), \quad (2.79)$$

$$\{\nabla, \bar{\Lambda}_\mu\} = \{\bar{\nabla}, \Lambda_\mu\} = 0, \quad (2.80)$$

and of the fermionic derivatives  $\nabla_A$  and the superfields  $(\Upsilon, \bar{\Upsilon})$ ,

$$\{\nabla_\mu, \Upsilon\} = -\frac{1}{2}\epsilon_{\mu\rho\sigma}[\nabla_{-\rho}, \nabla_{-\sigma}], \quad \{\bar{\nabla}_\mu, \bar{\Upsilon}\} = +\frac{1}{2}\epsilon_{\mu\rho\sigma}[\nabla_{-\rho}, \nabla_{-\sigma}], \quad (2.81)$$

$$\{\nabla, \Upsilon\} = +\mathcal{K} + \frac{i}{2}[\nabla_{+\rho}, \nabla_{-\rho}], \quad \{\bar{\nabla}, \bar{\Upsilon}\} = -\mathcal{K} + \frac{i}{2}[\nabla_{+\rho}, \nabla_{-\rho}], \quad (2.82)$$

$$\{\nabla, \bar{\Upsilon}\} = -\mathcal{G}, \quad \{\bar{\nabla}, \Upsilon\} = -\bar{\mathcal{G}}, \quad (2.83)$$

$$\{\nabla_\mu, \bar{\Upsilon}\} = \{\bar{\nabla}_\mu, \Upsilon\} = 0, \quad (2.84)$$

where we introduced the auxiliary superfield  $\mathcal{G}, \bar{\mathcal{G}}, \mathcal{K}$  whose first components are representing the bosonic auxiliary fields  $(G, \bar{G}, K)$  in the  $N = 4$   $D = 3$  twisted SYM multiplet,

$$\mathcal{G} = G + \dots, \quad \bar{\mathcal{G}} = \bar{G} + \dots, \quad \mathcal{K} = K + \dots. \quad (2.85)$$

One can show that all the other higher Jacobi identities can be expressed in terms of the fermionic covariant derivatives  $(\nabla, \bar{\nabla}_\mu, \nabla_\mu, \bar{\nabla})$ , the gauge covariant derivative superfields  $\nabla_{\pm\mu}$ , the non-vanishing fermionic superfields  $(\Upsilon, \bar{\Lambda}_\mu, \Lambda_\mu, \bar{\Upsilon})$  and the auxiliary superfields  $(\mathcal{G}, \bar{\mathcal{G}}, \mathcal{K})$ . As we will see, the lowest components of the superfields  $(\nabla_{\pm\mu}, \Upsilon, \bar{\Lambda}_\mu, \Lambda_\mu, \bar{\Upsilon}, \mathcal{G}, \bar{\mathcal{G}}, \mathcal{K})$  are representing the off-shell multiplet of  $N = 4$   $D = 3$  twisted SYM.

The  $SU(2)_E \times SU(2)_R$  rotational properties of the component fields can also be read off from the above Jacobi identities,

$$[J_\mu, \rho] = +\frac{1}{2}\lambda_\mu, \quad [J_\mu, \lambda_\nu] = -\frac{i}{2}\epsilon_{\mu\nu\rho}\lambda_\rho + \frac{1}{2}\delta_{\mu\nu}\rho, \quad (2.86)$$

$$[J_\mu, \bar{\rho}] = -\frac{1}{2}\lambda_\mu, \quad [J_\mu, \bar{\lambda}_\nu] = -\frac{i}{2}\epsilon_{\mu\nu\rho}\bar{\lambda}_\rho - \frac{1}{2}\delta_{\mu\nu}\bar{\rho}, \quad (2.87)$$

$$[R_\mu, \rho] = -\frac{1}{2}\lambda_\mu, \quad [R_\mu, \lambda_\nu] = -\frac{i}{2}\epsilon_{\mu\nu\rho}\lambda_\rho - \frac{1}{2}\delta_{\mu\nu}\rho, \quad (2.88)$$

$$[R_\mu, \bar{\rho}] = +\frac{1}{2}\lambda_\mu, \quad [R_\mu, \bar{\lambda}_\nu] = -\frac{i}{2}\epsilon_{\mu\nu\rho}\bar{\lambda}_\rho + \frac{1}{2}\delta_{\mu\nu}\bar{\rho}, \quad (2.89)$$

$$[J_\mu, G] = [J_\mu, \bar{G}] = [J_\mu, K] = [R_\mu, G] = [R_\mu, \bar{G}] = [R_\mu, K] = 0. \quad (2.90)$$

One sees that, after the twisting, the fermions  $(\rho, \bar{\lambda}_\mu, \lambda_\mu, \bar{\rho})$  are transforming as (scalar, vector, vector, scalar) while all of the auxiliary fields  $(G, \bar{G}, K)$  remain as scalars under  $J_\mu^{\text{diag}} = J_\mu + R_\mu$ .

The SUSY transformations of the component fields can be determined from the above Jacobi identities via

$$s_A \varphi = \{\nabla_A, \Psi\}|_{\theta'_s=0}, \quad (2.91)$$

where the  $\varphi$  denotes any of the component field  $(\mathcal{D}_{\pm\mu}, \rho, \bar{\lambda}_\mu, \lambda_\mu, \bar{\rho}, G, \bar{G}, K)$  in the SYM multiplet while the  $\Psi$  denotes the corresponding superfields  $(\nabla_{\pm\mu}, \Upsilon, \bar{\Lambda}_\mu, \Lambda_\mu, \bar{\Upsilon}, \mathcal{G}, \bar{\mathcal{G}}, \mathcal{K})$ , respectively. The symbol  $|_{\theta'_s=0}$  means that the  $\theta_A = (\theta, \bar{\theta}_\mu, \theta_\mu, \bar{\theta})$  are all taken to be zero. All the  $N = 4$   $D = 3$  twisted SUSY transformation laws for the component fields are listed in table 2. As a natural consequence of the constraints (2.38)–(2.41) or (2.52)–(2.55), the resulting  $N = 4$   $D = 3$  twisted SUSY algebra for the component fields closes off-shell modulo gauge transformations,

$$\{s, \bar{s}_\mu\}\varphi = -i[\mathcal{D}_{+\mu}, \varphi], \quad (2.92)$$

$$\{s_\mu, \bar{s}_\nu\}\varphi = -\epsilon_{\mu\nu\rho}[\mathcal{D}_{-\rho}, \varphi], \quad (2.93)$$

$$\{\bar{s}, s_\mu\}\varphi = -i[\mathcal{D}_{+\mu}, \varphi], \quad (2.94)$$

$$\{\text{others}\}\varphi = 0, \quad (2.95)$$

where the  $\varphi$  again denotes any component of the SYM multiplet  $(\mathcal{D}_{\pm\mu}, \rho, \bar{\lambda}_\mu, \lambda_\mu, \bar{\rho}, G, \bar{G}, K)$ . As described in [13, 14, 24], once all the SUSY transformation laws of the component fields are obtained, the corresponding superfield expressions can be given by operating  $e^{\delta_\theta}$  on the lowest components, where  $\delta_\theta = \theta s + \bar{\theta}_\mu \bar{s}_\mu + \theta_\mu s_\mu + \bar{\theta} \bar{s}$  for the twisted  $N = 4$   $D = 3$ ,

$$(\nabla_{\pm\mu}, \Upsilon, \bar{\Lambda}_\mu, \Lambda_\mu, \bar{\Upsilon}, \mathcal{G}, \bar{\mathcal{G}}, \mathcal{K}) = e^{\delta_\theta} (\mathcal{D}_{\pm\mu}, \rho, \bar{\lambda}_\mu, \lambda_\mu, \bar{\rho}, G, \bar{G}, K). \quad (2.96)$$

The  $N = 4$   $D = 3$  twisted SYM action can be manifestly constructed with the help of the “chiral” and “anti-chiral” superfields  $\nabla_{\pm\mu}$ . For example, if we focus on the  $\nabla_{\pm 3}$  which are subject to (2.67)–(2.68), the invariant action can be given by either of the following expressions,

$$\int d^3x \int d\bar{\theta}_1 d\bar{\theta}_2 d\theta_1 d\theta_2 \text{tr} \nabla_{+3} \nabla_{+3}, \quad \int d^3x \int d\bar{\theta} d\theta d\bar{\theta}_3 d\theta_3 \text{tr} \nabla_{-3} \nabla_{-3}. \quad (2.97)$$

	$s$	$\bar{s}_\mu$	$s_\mu$	$\bar{s}$
$\mathcal{D}_{+\nu}$	0	$-i\epsilon_{\mu\nu\rho}\lambda_\rho$	$+i\epsilon_{\mu\nu\rho}\bar{\lambda}_\rho$	0
$\mathcal{D}_{-\nu}$	$+\bar{\lambda}_\nu$	$-\delta_{\mu\nu}\rho$	$-\delta_{\mu\nu}\bar{\rho}$	$+\lambda_\nu$
$\rho$	$+K + \frac{i}{2}[\mathcal{D}_{+\rho}, \mathcal{D}_{-\rho}]$	0	$-\frac{1}{2}\epsilon_{\mu\rho\sigma}[\mathcal{D}_{-\rho}, \mathcal{D}_{-\sigma}]$	$-\bar{G}$
$\bar{\lambda}_\nu$	0	$-i[\mathcal{D}_{+\mu}, \mathcal{D}_{-\nu}]$ $+\delta_{\mu\nu}(K + \frac{i}{2}[\mathcal{D}_{+\rho}, \mathcal{D}_{-\rho}])$	$-\delta_{\mu\nu}G$	$-\frac{1}{2}\epsilon_{\nu\rho\sigma}[\mathcal{D}_{+\rho}, \mathcal{D}_{+\sigma}]$
$\lambda_\nu$	$+\frac{1}{2}\epsilon_{\nu\rho\sigma}[\mathcal{D}_{+\rho}, \mathcal{D}_{+\sigma}]$	$-\delta_{\mu\nu}\bar{G}$	$-i[\mathcal{D}_{+\mu}, \mathcal{D}_{-\nu}]$ $-\delta_{\mu\nu}(K - \frac{i}{2}[\mathcal{D}_{+\rho}, \mathcal{D}_{-\rho}])$	0
$\bar{\rho}$	$-G$	$+\frac{1}{2}\epsilon_{\mu\rho\sigma}[\mathcal{D}_{-\rho}, \mathcal{D}_{-\sigma}]$	0	$-K + \frac{i}{2}[\mathcal{D}_{+\rho}, \mathcal{D}_{-\rho}]$
$G$	0	$+\epsilon_{\mu\rho\sigma}[\mathcal{D}_{-\rho}, \bar{\lambda}_\sigma]$ $+i[\mathcal{D}_{+\mu}, \bar{\rho}]$	0	$+i[\mathcal{D}_{+\rho}, \bar{\lambda}_\rho]$
$\bar{G}$	$+i[\mathcal{D}_{+\rho}, \lambda_\rho]$	0	$-\epsilon_{\mu\rho\sigma}[\mathcal{D}_{-\rho}, \lambda_\sigma]$ $+i[\mathcal{D}_{+\mu}, \rho]$	0
$K$	$-\frac{i}{2}[\mathcal{D}_{+\rho}, \bar{\lambda}_\rho]$	$+\frac{1}{2}\epsilon_{\mu\rho\sigma}[\mathcal{D}_{-\rho}, \lambda_\sigma]$ $-\frac{i}{2}[\mathcal{D}_{+\mu}, \rho]$	$+\frac{1}{2}\epsilon_{\mu\rho\sigma}[\mathcal{D}_{-\rho}, \bar{\lambda}_\sigma]$ $+\frac{i}{2}[\mathcal{D}_{+\mu}, \bar{\rho}]$	$+\frac{i}{2}[\mathcal{D}_{+\rho}, \lambda_\rho]$

**Table 2:** SUSY trans. laws for twisted  $N = 4$   $D = 3$  SYM multiplet  $(\mathcal{D}_{\pm\mu}, \rho, \bar{\lambda}_\mu, \lambda_\mu, \bar{\rho}, G, \bar{G}, K)$

In terms of the lowest component fields  $\mathcal{D}_{\pm 3}$ , the above expressions are essentially equivalent to the successive operations of the supercharges on the lowest components  $\mathcal{D}_{\pm 3}$ ,

$$\int d^3x \int \bar{s}_1 \bar{s}_2 s_1 s_2 \text{tr} \mathcal{D}_{+3} \mathcal{D}_{+3}, \quad \int d^3x \int \bar{s} s \bar{s}_3 s_3 \text{tr} \mathcal{D}_{-3} \mathcal{D}_{-3}, \quad (2.98)$$

respectively. By consulting the SUSY transformation laws summarized in table 2, one can show that these two combinations are equivalent and give rise to the following SYM action,

$$S_{\text{TSYM}}^{N=4, D=3} = \int d^3x \frac{1}{2} \bar{s}_1 \bar{s}_2 s_1 s_2 \text{tr} \mathcal{D}_{+3} \mathcal{D}_{+3} = \int d^3x \frac{1}{2} \bar{s} s \bar{s}_3 s_3 \text{tr} \mathcal{D}_{-3} \mathcal{D}_{-3} \quad (2.99)$$

$$= \int d^3x \text{tr} \left[ \frac{1}{4} [\mathcal{D}_{+\mu}, \mathcal{D}_{-\mu}] [\mathcal{D}_{+\nu}, \mathcal{D}_{-\nu}] - \frac{1}{2} [\mathcal{D}_{+\mu}, \mathcal{D}_{+\nu}] [\mathcal{D}_{-\mu}, \mathcal{D}_{-\nu}] + K^2 + G\bar{G} \right. \\ \left. + i\bar{\lambda}_\mu [\mathcal{D}_{+\mu}, \rho] + i\lambda_\mu [\mathcal{D}_{+\mu}, \bar{\rho}] + \epsilon_{\mu\nu\rho} \lambda_\mu [\mathcal{D}_{-\nu}, \bar{\lambda}_\rho] \right]. \quad (2.100)$$

The exact form w.r.t. all the supercharges of twisted  $N = 4$   $D = 3$  manifestly ensures the invariance of the action under any of the SUSY transformation,

$$s_A S_{\text{TSYM}}^{N=4, D=3} = 0, \quad s_A = (s, \bar{s}_\mu, s_\mu, \bar{s}). \quad (2.101)$$

By substituting  $\mathcal{D}_{\pm\mu} = \partial_\mu - i(A_\mu \pm \phi^{(\mu)})$ , the action can be written as,

$$S_{\text{TSYM}}^{N=4, D=3} = \int d^3x \text{tr} \left[ \frac{1}{2} F_{\mu\nu} F_{\mu\nu} - [\mathcal{D}_\mu, \phi^{(\nu)}] [\mathcal{D}_\mu, \phi^{(\nu)}] - \frac{1}{2} [\phi^{(\mu)}, \phi^{(\nu)}] [\phi^{(\mu)}, \phi^{(\nu)}] + K^2 + G\bar{G} \right. \\ \left. + i\bar{\lambda}_\mu [\mathcal{D}_\mu, \rho] + i\lambda_\mu [\mathcal{D}_\mu, \bar{\rho}] + \epsilon_{\mu\nu\rho} \lambda_\mu [\mathcal{D}_\nu, \bar{\lambda}_\rho] \right. \\ \left. + \bar{\lambda}_\mu [\phi^{(\mu)}, \rho] + \lambda_\mu [\phi^{(\mu)}, \bar{\rho}] + i\epsilon_{\mu\nu\rho} \lambda_\mu [\phi^{(\nu)}, \bar{\lambda}_\rho] \right], \quad (2.102)$$

where  $F_{\mu\nu} \equiv i[\mathcal{D}_\mu, \mathcal{D}_\nu]$  are representing the field strength with the gauge covariant derivatives  $\mathcal{D}_\mu \equiv \partial_\mu - iA_\mu$ . The  $\text{tr}$  is representing the trace for the gauge group. One could notice that the kinetic terms, the potential terms and the Yukawa coupling terms are naturally arising from the combinations of the  $\mathcal{D}_{\pm\mu}$  in (2.100). The action (2.102) is the continuum counterpart of the twisted  $N = 4$   $D = 3$  lattice SYM action proposed in [6].

Here we have a couple of remarks regarding the action (2.102). First, the action can be expressed by re-writing the fermion kinetic terms and the Yukawa coupling terms in the following way,

$$S_{\text{TSYM}}^{N=4, D=3} = \int d^3x \text{tr} \left[ \frac{1}{2} F_{\mu\nu} F_{\mu\nu} - [\mathcal{D}_\mu, \phi^{(\nu)}][\mathcal{D}_\mu, \phi^{(\nu)}] - \frac{1}{2} [\phi^{(\mu)}, \phi^{(\nu)}][\phi^{(\mu)}, \phi^{(\nu)}] + K^2 + G\bar{G} + i\bar{\psi}_{i\alpha}(\gamma_\mu)_{\alpha\beta}[\mathcal{D}_\mu, \psi_{\beta i}] + \bar{\psi}_{i\alpha}[\phi^{(\mu)}, \psi_{\alpha j}](\gamma_\mu)_{ji} \right], \quad (2.103)$$

where we introduced the “untwisted” basis of fermions  $\psi_{\alpha i}$  and  $\bar{\psi}_{i\alpha}$  defined by the expansions w.r.t. the  $N = 4$   $D = 3$  twisted fermions  $(\rho, \lambda_\mu)$  and  $(\bar{\rho}, \bar{\lambda}_\mu)$ , respectively,

$$\psi_{\alpha i} = \frac{1}{\sqrt{3}}(\mathbf{1}\rho + \gamma_\mu \lambda_\mu)_{\alpha i}, \quad \bar{\psi}_{i\alpha} = \frac{1}{\sqrt{3}}(\mathbf{1}\bar{\rho} + \gamma_\mu \bar{\lambda}_\mu)_{i\alpha}, \quad (2.104)$$

The  $\mathbf{1}$  again denotes a two-by-two unit matrix while the gamma matrices  $\gamma_\mu$  are taken as the Pauli matrices,  $\gamma_\mu (\mu = 1, 2, 3) = (\sigma_1, \sigma_2, \sigma_3)$ . In terms of the untwisted fermions (2.104), one could see the manifest invariance of the action (2.103) under the independent  $\text{SU}(2)_E$  and  $\text{SU}(2)_R$  rotations. The expressions (2.104) also imply that the  $N = 4$   $D = 3$  twisted fermions have one-to-one correspondences with a three dimensional Dirac-Kähler fermion components [25]. This is more clearly seen in the corresponding lattice formulation given in [6], where the  $N = 4$   $D = 3$  twisted fermions  $(\rho, \bar{\lambda}_\mu, \lambda_\mu, \bar{\rho})$  are essentially embedded in the three dimensional lattice as a (0-form, 1-form, 2-form, 3-form), respectively.

One should also notice from the second equality in (2.99) that the  $N = 4$   $D = 3$  twisted SYM action can be expressed by the  $s\bar{s}$ -exact form,

$$S_{\text{TSYM}}^{N=4, D=3} = \frac{1}{2} s\bar{s} \int d^3x \bar{s}_3 s_3 \text{tr} \mathcal{D}_{-3} \mathcal{D}_{-3} \quad (2.105)$$

$$= \frac{1}{2} s\bar{s} \int d^3x \text{tr} \left[ -\frac{1}{3} \epsilon_{\mu\nu\rho} \mathcal{D}_{-\mu} [\mathcal{D}_{-\nu}, \mathcal{D}_{-\rho}] + 2\rho\bar{\rho} \right]. \quad (2.106)$$

After using the cyclic trace property, partial integrations and the commuting nature of the derivative operators,  $[\partial_\mu, \partial_\nu] = 0$ , one finds that the above action can be written down as the  $s\bar{s}$ -exact form on the following Chern-Simons type action attached with the fermion bilinear term  $\rho\bar{\rho}$ ,

$$S_{\text{TSYM}}^{N=4, D=3} = \frac{1}{2} s\bar{s} S_{\text{SCS}}, \quad (2.107)$$

$$S_{\text{SCS}} \equiv \int d^3x \text{tr} \left[ \epsilon_{\mu\nu\rho} (A_\mu^- [\partial_\nu, A_\rho^-] - \frac{i}{3} A_\mu^- [A_\nu^-, A_\rho^-]) + 2\rho\bar{\rho} \right], \quad (2.108)$$

where the symbols  $A_\mu^-$  are defined by  $A_\mu^- \equiv A_\mu - \phi^{(\mu)}$ . It is rather striking to recognize that the SYM action and the above type of Chern-Simons action are intrinsically related

by the twisted SUSY transformations. This result is actually consistent with the (on-shell) formulations of  $N = 4$   $D = 3$  twisted SYM with the auxiliary one-forms  $B$  and  $\bar{B}$  given in [17, 18]. Furthermore, thanks to the manifestly off-shell formulation here, it is easy to show that the  $S_{\text{SCS}}$  can also be expressed by the  $\bar{s}_1 s_1$  and  $\bar{s}_2 s_2$  exact forms,

$$S_{\text{SCS}} = \int d^3x \bar{s}_3 s_3 \text{tr} \mathcal{D}_{-3} \mathcal{D}_{-3} = \int d^3x \bar{s}_1 s_1 \text{tr} \mathcal{D}_{-1} \mathcal{D}_{-1} = \int d^3x \bar{s}_2 s_2 \text{tr} \mathcal{D}_{-2} \mathcal{D}_{-2}, \quad (2.109)$$

from which it obeys the invariance of the  $S_{\text{SCS}}$  under the six of the twisted SUSY transformations  $(s_1, s_2, s_3, \bar{s}_1, \bar{s}_2, \bar{s}_3)$ ,

$$s_\mu S_{\text{SCS}} = \bar{s}_\mu S_{\text{SCS}} = 0, \quad (\mu = 1, 2, 3). \quad (2.110)$$

It is important to recognize here that the supercharges  $(s_\mu, \bar{s}_\mu)$  and the sub-multiplet  $(\mathcal{D}_{-\mu}, \rho, \bar{\rho})$  form an off-shell closed sub-algebra embedded in the entire  $N = 4$   $D = 3$  twisted SUSY algebra (2.92)–(2.95),

$$\{s_\mu, \bar{s}_\nu\} \varphi_{\text{sub}} = -\epsilon_{\mu\nu\rho} [\mathcal{D}_{-\rho}, \varphi_{\text{sub}}], \quad (2.111)$$

$$\{s_\mu, s_\nu\} \varphi_{\text{sub}} = \{\bar{s}_\mu, \bar{s}_\nu\} \varphi_{\text{sub}} = 0, \quad (2.112)$$

where  $\varphi_{\text{sub}}$  is representing any component of the sub-multiplet  $\varphi_{\text{sub}} = (\mathcal{D}_{-\mu}, \rho, \bar{\rho})$ .

Another important observation is that the Lagrangian density in the  $S_{\text{SCS}}$  transforms as a scalar only under the twisted rotations  $J_\mu^{\text{diag}} = J_\mu + R_\mu$  and not under the  $J_\mu$  and the  $R_\mu$  independently, which implies that the action  $S_{\text{SCS}}$  is purely a twisted object. As a consequence, one cannot perform “untwisting” the action  $S_{\text{SCS}}$  even in the flat spacetime. This should be compared to the rotational property of  $S_{\text{TSYM}}^{N=4, D=3}$  which is invariant under the  $J_\mu$  and  $R_\mu$  rotations independently. The other type of  $J_\mu^{\text{diag}}$  scalar Chern-Simons type action can be found by noticing that the  $S_{\text{TSYM}}^{N=4, D=3}$  is also expressed as the following form,

$$S_{\text{TSYM}}^{N=4, D=3} = \frac{1}{6} \int d^3x \bar{s}_\mu s_\mu s \bar{s} \text{tr} \mathcal{D}_{-\nu} \mathcal{D}_{-\nu} \quad (2.113)$$

$$\equiv \frac{1}{2} s_\mu \bar{s}_\mu S'_{\text{SCS}}, \quad (2.114)$$

where  $\mu$  and  $\nu$  are summed up from 1 to 3. The  $S'_{\text{SCS}}$  is given by the Chern-Simons type action with a mixed combination of  $A_\mu^\pm \equiv A_\mu \pm \phi^{(\mu)}$  attached with another fermion bilinear terms  $\lambda_\mu \bar{\lambda}_\mu$ ,

$$S'_{\text{SCS}} = -\frac{1}{3} \int d^3x s \bar{s} \text{tr} \mathcal{D}_{-\nu} \mathcal{D}_{-\nu} \quad (2.115)$$

$$= \int d^3x \text{tr} \left[ -\frac{1}{3} \epsilon_{\mu\nu\rho} \mathcal{D}_{-\mu} [\mathcal{D}_{+\nu}, \mathcal{D}_{+\rho}] + \frac{2}{3} \lambda_\mu \bar{\lambda}_\mu \right] \quad (2.116)$$

$$= \int d^3x \text{tr} \left[ \epsilon_{\mu\nu\rho} \left\{ \frac{1}{3} (A_\mu^+ + 2A_\mu^-) [\partial_\nu, A_\rho^+] - \frac{i}{3} A_\mu^- [A_\nu^+, A_\rho^+] \right\} + \frac{2}{3} \lambda_\mu \bar{\lambda}_\mu \right], \quad (2.117)$$

It is clearly seen from its exact form that the  $S'_{\text{SCS}}$  is invariant under the scalar type twisted SUSY transformations  $s$  and  $\bar{s}$ ,

$$s S'_{\text{SCS}} = \bar{s} S'_{\text{SCS}} = 0. \quad (2.118)$$

The above SUSY invariance is again supported by the existence of the sub-algebra and the sub-multiplet embedded in the entire twisted  $N = 4$   $D = 3$  algebra (2.92)–(2.95),

$$\{s, \bar{s}\} \varphi'_{\text{sub}} = s^2 \varphi'_{\text{sub}} = \bar{s}^2 \varphi'_{\text{sub}} = 0, \quad (2.119)$$

which holds off-shell for the sub-multiplet  $\varphi'_{\text{sub}} = (\mathcal{D}_{\pm\mu}, \lambda_\mu, \bar{\lambda}_\mu)$ .

Remembering that the  $N = 4$   $D = 3$  twisted SYM action  $S_{\text{TSYM}}^{N=4, D=3}$  can be realized on the lattice consistently with the lattice Leibniz rule conditions [6], one may wonder the above exactness relation between the SYM and the super Chern-Simons would be playing a key role also in realizing the Chern-Simons on the lattice. Since addressing this topic is beyond the initial scope of this paper, we keep this subject as our future study. In the next section we will shed light on the rather different aspect of the  $N = 4$   $D = 3$  twisted SYM, namely its dimensional reduction aspect from  $N = 2$   $D = 4$  twisted SYM.

### 3. Two possible twists of $N = 4$ $D = 3$ SYM

In this section, by a dimensional reduction of the  $N = 2$   $D = 4$  twisted SYM constraints, we explore the two possible twists of  $N = 4$   $D = 3$  SYM and we show that the  $N = 4$   $D = 3$  twisted SYM in the last section is essentially classified as the B-type twisted SYM.

The formulations of the  $N = 2$   $D = 4$  twisted SYM in terms of the superconnection method are given in [14, 23]. In [14], the detailed analysis originated from the  $N = D = 4$  Dirac-Kähler point of view is also explicitly elaborated. We start from the following  $N = 2$   $D = 4$  twisted SYM constraints in the twisted superspace  $(x, \theta^+, \theta_\mu^+, \theta_{\rho\sigma}^+)$  [14],

$$\{\nabla^+, \nabla_\mu^+\} = -i\nabla_{\underline{\mu}}, \quad \{\nabla_{\rho\sigma}^+, \nabla_\mu^+\} = +i\delta_{\rho\sigma\mu\nu}^+ \nabla_{\underline{\nu}}, \quad (3.1)$$

$$\{\nabla^+, \nabla^+\} = -iW, \quad \{\nabla_{\mu\nu}^+, \nabla_{\rho\sigma}^+\} = -i\delta_{\mu\nu\rho\sigma}^+ W, \quad (3.2)$$

$$\{\nabla_\mu^+, \nabla_\nu^+\} = -i\delta_{\mu\nu} F, \quad \{\text{others}\} = 0, \quad (3.3)$$

where the symbol  $\delta_{\mu\nu\rho\sigma}^+$  is defined as  $\delta_{\mu\nu\rho\sigma}^+ \equiv \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma}$ . The second rank tensor  $\nabla_{\mu\nu}^+$  satisfies the self-duality condition,  $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\nabla_{\rho\sigma}^+ = \nabla_{\mu\nu}^+$ . The symbols  $(\nabla^+, \nabla_\mu^+, \nabla_{\rho\sigma}^+)$  are denoting the supergauge covariant derivatives which consist of the  $N = 2$   $D = 4$  superderivatives  $(D^+, D_\mu^+, D_{\rho\sigma}^+)$  and the superconnections  $(\Gamma^+, \Gamma_\mu^+, \Gamma_{\rho\sigma}^+)$ ,

$$\nabla^+ = D^+ - i\Gamma^+(\theta^+, \theta_\alpha^+, \theta_{\gamma\delta}^+), \quad (3.4)$$

$$\nabla_\mu^+ = D_\mu^+ - i\Gamma_\mu^+(\theta^+, \theta_\alpha^+, \theta_{\gamma\delta}^+), \quad (3.5)$$

$$\nabla_{\rho\sigma}^+ = D_{\rho\sigma}^+ - i\Gamma_{\rho\sigma}^+(\theta^+, \theta_\alpha^+, \theta_{\gamma\delta}^+), \quad (3.6)$$

where the superderivatives  $(D^+, D_\mu^+, D_{\rho\sigma}^+)$  satisfy the following  $N = 2$   $D = 4$  twisted SUSY algebra,

$$\{D^+, D_\mu^+\} = -i\partial_\mu, \quad \{D_{\rho\sigma}^+, D_\mu^+\} = +i\delta_{\rho\sigma\mu\nu}^+ \partial_\nu, \quad \{\text{others}\} = 0. \quad (3.7)$$

We denote the expansions of the bosonic gauge covariant superfields  $(\nabla_{\underline{\mu}}^+, W, F)$  in the r.h.s. of (3.1)–(3.3) as,

$$\nabla_{\underline{\mu}} = \partial_\mu - iA_\mu + \dots, \quad W = A + \dots, \quad F = B + \dots, \quad (3.8)$$

where the  $A_\mu$ ,  $A$  and  $B$  are representing the four dimensional gauge field and the two independent scalar fields, respectively. All the component fields in the  $\nabla_\mu$ ,  $W$  and  $F$  can be expressed by the combinations of the component fields embedded in the superconnections  $(\Gamma^+, \Gamma_\mu^+, \Gamma_{\rho\sigma}^+)$  subject to the constraints (3.1)–(3.3). The dots in (3.8) denote the possible  $\theta_A^+ = (\theta^+, \theta_\mu^+, \theta_{\rho\sigma}^+)$  expansion terms.

The  $SO(4)_E \times SU(2)_R$  rotational properties of the (super)covariant derivatives are given by [14],

$$[J_{\mu\nu}^+, \nabla^+] = +\frac{i}{2}\nabla_{\mu\nu}^+, \quad [J_{\mu\nu}^-, \nabla_\rho^+] = -\frac{i}{2}\delta_{\mu\nu\rho\sigma}^- \nabla_\sigma^+, \quad (3.9)$$

$$[J_{\mu\nu}^+, \nabla_{\rho\sigma}^+] = -\frac{i}{2}\delta_{\mu\nu\rho\sigma}^+ \nabla^+ + \frac{i}{4}(\delta_{\mu\nu\rho\lambda}^+ \nabla_{\sigma\lambda}^+ - \delta_{\mu\nu\sigma\lambda}^+ \nabla_{\rho\lambda}^+), \quad (3.10)$$

$$[J_{\mu\nu}^-, \nabla^+] = [J_{\mu\nu}^+, \nabla_\rho^+] = [J_{\mu\nu}^-, \nabla_{\rho\sigma}^+] = 0, \quad (3.11)$$

$$[J_{\mu\nu}^+, \nabla_{\underline{\rho}}] = -\frac{i}{2}\delta_{\mu\nu\rho\sigma}^+ \nabla_{\underline{\sigma}}, \quad [J_{\mu\nu}^-, \nabla_{\underline{\rho}}] = -\frac{i}{2}\delta_{\mu\nu\rho\sigma}^- \nabla_{\underline{\sigma}}, \quad (3.12)$$

$$[J_{\mu\nu}^+, W] = [J_{\mu\nu}^-, W] = [J_{\mu\nu}^+, F] = [J_{\mu\nu}^-, F] = 0, \quad (3.13)$$

$$[R_{\mu\nu}^+, \nabla^+] = -\frac{i}{2}\nabla_{\mu\nu}^+, \quad [R_{\mu\nu}^+, \nabla_\rho^+] = -\frac{i}{2}\delta_{\mu\nu\rho\sigma}^- \nabla_\sigma^+, \quad (3.14)$$

$$[R_{\mu\nu}^+, \nabla_{\rho\sigma}^+] = +\frac{i}{2}\delta_{\mu\nu\rho\sigma}^+ \nabla^+ + \frac{i}{4}(\delta_{\mu\nu\rho\lambda}^+ \nabla_{\sigma\lambda}^+ - \delta_{\mu\nu\sigma\lambda}^+ \nabla_{\rho\lambda}^+), \quad (3.15)$$

$$[R_{\mu\nu}^-, \nabla^+] = [R_{\mu\nu}^-, \nabla_\rho^+] = [R_{\mu\nu}^-, \nabla_{\rho\sigma}^+] = 0, \quad (3.16)$$

$$[R_{\mu\nu}^+, \nabla_{\underline{\rho}}] = [R_{\mu\nu}^+, W] = [R_{\mu\nu}^+, F] = 0, \quad (3.17)$$

where  $J_{\mu\nu}^+$  and  $J_{\mu\nu}^-$  denote the self-dual and anti-selfdual part of  $SO(4)_E$  Euclidean Lorentz generators while the  $R_{\mu\nu}^+$  denote  $SU(2)_R$  internal rotation generators. Note that  $R_{\mu\nu}^+$  is also subject to the self-duality condition,  $R_{\mu\nu}^+ = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} R_{\rho\sigma}^+$ . The symbols  $\delta_{\mu\nu\rho\sigma}^\pm \equiv \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} \pm \epsilon_{\mu\nu\rho\sigma}$  are projecting the self-dual and anti-selfdual part, respectively.

Now we perform the dimensional reduction of the  $N = 2$   $D = 4$  twisted SYM constraints to the  $N = 4$   $D = 3$  constraints. We take the component fields independent of the fourth direction  $x_4$  and denote the (super)gauge covariant derivatives in terms of the following  $N = 4$   $D = 3$  notations,

$$\nabla^+ \rightarrow \nabla^1, \quad \nabla_1^+ \rightarrow \nabla_1^2, \quad \nabla_2^+ \rightarrow \nabla_2^2, \quad \nabla_3^+ \rightarrow \nabla_3^2, \quad \nabla_4^+ \rightarrow -\nabla^2, \quad (3.18)$$

$$\nabla_{12}^+ \rightarrow \nabla_3^1, \quad \nabla_{13}^+ \rightarrow -\nabla_2^1, \quad \nabla_{14}^+ \rightarrow \nabla_1^1, \quad \nabla_{\underline{4}}^+ \rightarrow -G, \quad (3.19)$$

where the  $G$  is representing the scalar field originated from the gauge field in fourth dimension. By the dimensional reduction, the original Euclidean rotational group  $SO(4)_E$  is reduced into  $SU(2)_E$  which is the covering group of three dimensional Euclidean rotation  $SO(3)_E$ , while the original internal symmetry  $SU(2)_R$  remains intact. Furthermore, as is pointed out in [26], we have yet another  $SU(2)$  symmetry associated with the  $N = 4$   $D = 3$  SUSY algebra, which is denoted as  $SU(2)_N$ . The existence of two independent internal symmetries  $SU(2)_R$  and  $SU(2)_N$  leads to the two possible topological twists [17, 18] which we will explicitly see in the following.



In terms of three dimensional notation (3.18)–(3.19), the constraints (3.1)–(3.3) turn into the following form of  $N = 4$   $D = 3$  SYM constraints,

$$\{\nabla^a, \nabla_\mu^b\} = -i\epsilon^{ab}\nabla_\mu, \quad \{\nabla_\mu^a, \nabla_\nu^b\} = +i\epsilon^{ab}\epsilon_{\mu\nu\rho}\nabla_\rho - i\delta_{\mu\nu}\phi^{ab}, \quad (3.20)$$

$$\{\nabla^a, \nabla^b\} = -i\phi^{ab}, \quad \{\text{others}\} = 0, \quad (3.21)$$

where the subscripts  $\mu, \nu, \rho$  run from 1 to 3 while the  $SU(2)_N$  superscripts  $a, b$  take 1 or 2. The scalar fields  $\phi^{ab}$  form the triplet state of  $SU(2)_N$  where each component of  $\phi^{ab}$  is defined by the scalar fields introduced in (3.2), (3.3) and (3.19),

$$\phi^{11} = W, \quad \phi^{12} = \phi^{21} = G, \quad \phi^{22} = F. \quad (3.22)$$

The transformations of the supercovariant derivatives under the whole symmetry group  $SU(2)_E \times SU(2)_R \times SU(2)_N$  are given by

$$[J_\mu, \nabla^a] = +\frac{i}{2}\nabla_\mu^a, \quad [J_\mu, \nabla_\nu^a] = -\frac{i}{2}\epsilon_{\mu\nu\rho}\nabla_\rho^a - \frac{i}{2}\delta_{\mu\nu}\nabla^a, \quad (3.23)$$

$$[R_\mu, \nabla^a] = -\frac{i}{2}\nabla_\mu^a, \quad [R_\mu, \nabla_\nu^a] = -\frac{i}{2}\epsilon_{\mu\nu\rho}\nabla_\rho^a + \frac{i}{2}\delta_{\mu\nu}\nabla^a, \quad (3.24)$$

$$[N_\mu, \nabla_A^a] = \frac{1}{2}(\gamma_\mu)^{ab}\nabla_A^a, \quad (3.25)$$

where  $J_\mu$ ,  $R_\mu$  and  $N_\mu$  denote the generators of  $SU(2)_E$ ,  $SU(2)_R$  and  $SU(2)_N$ , respectively.<sup>3</sup> In the last line,  $\nabla_A^a$  represents any of  $(\nabla^a, \nabla_\mu^a)$ . The gamma matrices  $\gamma_\mu$  are taken as the Pauli matrices,  $\gamma_\mu (\mu = 1, 2, 3) = (\sigma_1, \sigma_2, \sigma_3)$ . The generators  $J_\mu$ ,  $R_\mu$  and  $N_\mu$  obey the independent  $SU(2)$  algebra,

$$[J_\mu, J_\nu] = -i\epsilon_{\mu\nu\rho}J_\rho, \quad [R_\mu, R_\nu] = -i\epsilon_{\mu\nu\rho}R_\rho, \quad [N_\mu, N_\nu] = -i\epsilon_{\mu\nu\rho}N_\rho, \quad (3.26)$$

$$[J_\mu, R_\nu] = [J_\mu, N_\nu] = [R_\mu, N_\nu] = 0. \quad (3.27)$$

The first topological twist of  $N = 4$   $D = 3$ , which is called the A-type or the super BF type twist in the literatures, is given by taking the diagonal subgroup of  $SU(2)_E \times SU(2)_R$  [17, 18]. After the twisting, the entire rotational symmetries are governed by the twisted Lorentz rotation  $(SU(2)_E \times SU(2)_R)_{\text{diag}}$  generated by  $J_\mu^{\text{diag}} \equiv J_\mu + R_\mu$  and the internal rotation  $SU(2)_N$  by  $N_\mu$ ,

$$[J_\mu^{\text{diag}}, \nabla^a] = 0, \quad [J_\mu^{\text{diag}}, \nabla_\nu^a] = -i\epsilon_{\mu\nu\rho}\nabla_\rho^a \quad [N_\mu, \nabla_A^a] = \frac{1}{2}(\gamma_\mu)^{ab}\nabla_A^a, \quad (3.28)$$

namely,  $\nabla^a$  and  $\nabla_\mu^a$  are transforming as,

$$\nabla^a : (\mathbf{1}, \mathbf{2}), \quad \nabla_\mu^a : (\mathbf{3}, \mathbf{2}), \quad (3.29)$$

of  $(SU(2)_E \times SU(2)_R)_{\text{diag}} \times SU(2)_N$ . It is also easy to see from the constraints (3.20) and (3.21) that the gauge field  $A_\mu$  and the scalar fields are transforming as,

$$A_\mu : (\mathbf{3}, \mathbf{1}), \quad \phi^{ab} : (\mathbf{1}, \mathbf{3}). \quad (3.30)$$

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<sup>3</sup>The  $J_\mu$  and  $R_\mu$  are defined by  $J_1 = J_{14}^+ + J_{14}^-$ ,  $J_2 = -J_{13}^+ - J_{13}^-$ ,  $J_3 = J_{12}^+ + J_{12}^-$ ,  $R_1 = R_{14}^+ + R_{14}^-$ ,  $R_2 = -R_{13}^+ - R_{13}^-$  and  $R_3 = R_{12}^+ + R_{12}^-$  in terms of  $N = 2$   $D = 4$  notation.

Notice that the  $\phi^{ab}$  still transform as scalars after the A-type twist. By analyzing the Jacobi identities together with the constraints (3.20) and (3.21), one can construct the corresponding off-shell SYM multiplet which consists of the gauge field  $A_\mu$ , the scalar fields  $\phi^{ab}$  as well as the twisted fermions  $\rho^a$ ,  $\lambda_\mu^a$  and the bosonic auxiliary field  $H_\mu$  transforming as,

$$\rho^a : (\mathbf{1}, \mathbf{2}), \quad \lambda_\mu^a : (\mathbf{3}, \mathbf{2}), \quad H_\mu : (\mathbf{3}, \mathbf{1}). \quad (3.31)$$

The SUSY transformations of the component fields and the corresponding SYM action of the A-type twist can also be obtained through the Jacobi identity analyses under the constraints (3.20)–(3.21) just as in the previous section.

The second topological twist of  $N = 4$   $D = 3$ , which is called the B-type or the Blau-Thompson type twist, is given by taking the diagonal subgroup of  $SU(2)_E \times SU(2)_N$  [17, 18]. One sees that the basis of  $\nabla_A^a$  appeared in (3.20) and (3.21) is not appropriate for the B-type twist since the operations of  $J_\mu$  and  $N_\mu$  in (3.23) and (3.25) are not on the same footing. One of the appropriate basis for the B-type twist could be found after taking the following linear combinations of the super gauge covariant derivatives,

$$\nabla'^1 \equiv +\frac{1}{2} (\nabla^1 + i\nabla_1^1 + \nabla_2^2 + i\nabla_3^2), \quad \nabla'^2 \equiv +\frac{1}{2} (\nabla^2 - i\nabla_1^2 - \nabla_2^1 + i\nabla_3^1), \quad (3.32)$$

$$\nabla_1'^1 \equiv -\frac{i}{2} (\nabla^1 + i\nabla_1^1 - \nabla_2^2 - i\nabla_3^2), \quad \nabla_1'^2 \equiv +\frac{i}{2} (\nabla^2 - i\nabla_1^2 + \nabla_2^1 - i\nabla_3^1), \quad (3.33)$$

$$\nabla_2'^1 \equiv -\frac{1}{2} (\nabla^2 + i\nabla_1^2 - \nabla_2^1 - i\nabla_3^1), \quad \nabla_2'^2 \equiv +\frac{1}{2} (\nabla^1 - i\nabla_1^1 + \nabla_2^2 - i\nabla_3^2), \quad (3.34)$$

$$\nabla_3'^1 \equiv -\frac{i}{2} (\nabla^2 + i\nabla_1^2 + \nabla_2^1 + i\nabla_3^1), \quad \nabla_3'^2 \equiv -\frac{i}{2} (\nabla^1 - i\nabla_1^1 - \nabla_2^2 + i\nabla_3^2), \quad (3.35)$$

and making the slight re-definitions of  $R_\mu$  and  $N_\mu$ ,

$$R'_1 \equiv -R_3, \quad R'_2 \equiv -R_2, \quad R'_3 \equiv -R_1, \quad (3.36)$$

$$N'_1 \equiv -N_3, \quad N'_2 \equiv -N_2, \quad N'_3 \equiv -N_1. \quad (3.37)$$

We then see that the relations (3.20)–(3.21) are re-expressed as

$$\{\nabla'^a, \nabla_\mu'^b\} = -i\epsilon^{ab} (\nabla_\mu + \phi^{(\mu)}), \quad (3.38)$$

$$\{\nabla_\mu'^a, \nabla_\nu'^b\} = +i\epsilon^{ab} \epsilon_{\mu\nu\rho} (\nabla_\rho - \phi^{(\rho)}), \quad \{\text{others}\} = 0, \quad (3.39)$$

where the  $\phi^{(\mu)}$  are given in terms of  $W$ ,  $F$  and  $G$ ,

$$\phi^{(1)} \equiv -iG, \quad \phi^{(2)} \equiv +\frac{1}{2}(W + F), \quad \phi^{(3)} \equiv -\frac{i}{2}(W - F). \quad (3.40)$$

Accordingly, the relations (3.23)–(3.25) are also re-expressed by

$$[J_\mu, \nabla'^a] = +\frac{i}{2} \nabla_\mu'^a, \quad [J_\mu, \nabla_\nu'^a] = -\frac{i}{2} \epsilon_{\mu\nu\rho} \nabla_\rho'^a - \frac{i}{2} \delta_{\mu\nu} \nabla'^a, \quad (3.41)$$

$$[N'_\mu, \nabla'^a] = -\frac{i}{2} \nabla_\mu'^a, \quad [N'_\mu, \nabla_\nu'^a] = -\frac{i}{2} \epsilon_{\mu\nu\rho} \nabla_\rho'^a + \frac{i}{2} \delta_{\mu\nu} \nabla'^a, \quad (3.42)$$

$$[R'_\mu, \nabla_A'^a] = \frac{1}{2} (\gamma_\mu)^{ab} \nabla_A'^a. \quad (3.43)$$

Notice that, after the re-definitions (3.32)–(3.35), the  $SU(2)_E$  and the  $SU(2)_N$  are operating on the same footing as if the role of the  $SU(2)_N$  and the  $SU(2)_R$  were interchanged. The second topological twist can be appropriately performed on this basis by taking the diagonal subgroup of  $SU(2)_E \times SU(2)_N$ . After the B-type twist, the rotational symmetries are governed by the generators  $J_\mu^{\text{diag}} \equiv J_\mu + N'_\mu$  and  $R'_\mu$  which are representing the twisted Lorentz of  $(SU(2)_E \times SU(2)_N)_{\text{diag}}$  and the internal rotation generators of  $SU(2)_R$ , respectively. From (3.41)–(3.43), one could obviously see

$$[J_\mu^{\text{diag}}, \nabla'^a] = 0, \quad [J_\mu^{\text{diag}}, \nabla'_\nu{}^a] = -i\epsilon_{\mu\nu\rho} \nabla'_\rho{}^a, \quad [R'_\mu, \nabla'_A{}^a] = \frac{1}{2}(\gamma_\mu)^{ab} \nabla'_A{}^a. \quad (3.44)$$

Namely the  $\nabla'^a$  and  $\nabla'_\mu{}^a$  are transforming as,

$$\nabla'^a : (\mathbf{1}, \mathbf{2}), \quad \nabla'_\mu{}^a : (\mathbf{3}, \mathbf{2}), \quad (3.45)$$

of  $(SU(2)_E \times SU(2)_N)_{\text{diag}} \times SU(2)_R$ . The gauge fields  $A_\mu$  and the scalar fields  $\phi^{(\mu)}$  are accordingly transforming as

$$A_\mu : (\mathbf{3}, \mathbf{1}), \quad \phi^{(\mu)} : (\mathbf{3}, \mathbf{1}). \quad (3.46)$$

Notice that, after the B-twist, the scalar fields  $\phi^{(\mu)}$  transform as a three dimensional vector just like the gauge fields. The B-type twisted fermions  $(\rho^a, \lambda_\mu^a)$  are transforming in a similar way as the A-type twisted ones while the auxiliary field  $H^{ab}$  transform as a  $SU(2)_R$  triplet states,

$$\rho^a : (\mathbf{1}, \mathbf{2}), \quad \lambda_\mu^a : (\mathbf{3}, \mathbf{2}), \quad H^{ab} : (\mathbf{1}, \mathbf{3}). \quad (3.47)$$

It comes clear from the constraints (3.38)–(3.39) and the representations of the components (3.45)–(3.47) that the  $N = 4$   $D = 3$  twisted SYM in section 2 can be essentially identified as the B-type twisted SYM described in this section. We have the following notational identifications for the  $SU(2)_R$  doublet and triplet states,

$$(\nabla_A, \bar{\nabla}_A) \leftrightarrow \nabla'_A{}^a, \quad (\rho, \bar{\rho}) \leftrightarrow \rho^a, \quad (\lambda_\mu, \bar{\lambda}_\mu) \leftrightarrow \lambda_\mu^a, \quad (G, \bar{G}, K) \leftrightarrow H^{ab}, \quad (3.48)$$

where the appropriate sign re-definitions are understood. Note that the internal symmetry generators  $R'_\mu$  in section 2 can be identified as  $N'_\mu$  in this section. The above correspondences indicate that the lattice formulation of  $N = 4$   $D = 3$  twisted SYM given in [6] is essentially classified as the B-type twisted SYM.

#### 4. The twisted $N = 2$ $D = 4$ and $N = 4$ $D = 3$ SYM from the lattice point of view

Knowing the classification of the two inequivalent topological twists of  $N = 4$   $D = 3$  SYM and remembering the lattice realization of B-type twisted SYM [6], one should ask the possibility of formulating the A-type twisted SYM on the lattice as well. In this section, we consider the lattice Leibniz rule and the lattice gauge covariance for the  $N = 2$   $D = 4$  and  $N = 4$   $D = 3$  twisted SUSY algebra and multiplet. We then explicitly see that the

B-type SYM is only the case which can be consistently realized on the three dimensional lattice satisfying these criteria.

Let us briefly remind the basic idea of the lattice Leibniz rule introduced in [4, 5]. The importance of Leibniz rule is also stressed in the context of the non-commutative differential geometry on the lattice [27]. Since on the lattice there are no infinitesimal translations, the derivative operators should be replaced by the corresponding difference operators of either forward or backward,  $P_\mu = i\partial_\mu \rightarrow i\Delta_{\pm\mu}$ . The operation of the difference operators on the lattice is naturally defined by the following type of “shifted” commutators,

$$(\Delta_{\pm\mu}\Phi(x)) = \Delta_{\pm\mu}\Phi(x) - \Phi(x \pm n_\mu)\Delta_{\pm\mu}, \tag{4.1}$$

where the  $n_\mu$  denote three dimensional lattice unit vectors. The  $\Delta_{\pm\mu}$  are located on links from  $x$  to  $x \pm n_\mu$ , respectively, and taking the unit values for the generic site  $x$ ,

$$\Delta_{\pm\mu} = (\Delta_{\pm\mu})_{x \pm n_\mu, x} = \mp 1. \tag{4.2}$$

Correspondingly, we define the lattice supercharges  $Q_A$  on the links from  $x$  to  $x + a_A$  whose operations are defined by the “shifted” (anti)commutators,

$$(Q_A\Phi(x)) = (Q_A)_{x+a_A, x}\Phi(x) - (-)^{|\Phi|}\Phi(x + a_A)(Q_A)_{x+a_A, x}, \tag{4.3}$$

where the symbol  $|\Phi|$  takes the value of 0 or 1 for the bosonic or the fermionic  $\Phi$ , respectively. Since the supercharges  $Q_A$  are located on links, the anti-commutators of supercharges are naturally defined by the successive connections of link supercharges,

$$\{Q_A, Q_B\}_{x+a_A+a_B, x} = (Q_A)_{x+a_A+a_B, x+a_B}(Q_B)_{x+a_B, x} + (Q_B)_{x+a_A+a_B, x+a_A}(Q_A)_{x+a_A, x} \tag{4.4}$$

In terms of these ingredients, the lattice SUSY algebra can be expressed as

$$\{Q_A, Q_B\}_{x+a_A+a_B, x} = (\Delta_{\pm\mu})_{x, x \pm n_\mu} \tag{4.5}$$

provided the following lattice Leibniz rule conditions hold (see figure 1 and figure 2),

$$a_A + a_B = +n_\mu \quad \text{for} \quad \Delta_{+\mu}, \tag{4.6}$$

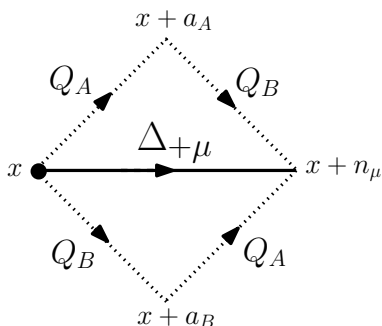
$$a_A + a_B = -n_\mu \quad \text{for} \quad \Delta_{-\mu}. \tag{4.7}$$

It has been pointed out that the Dirac-Kähler twisted  $N = D = 2$ ,  $N = 4$   $D = 3$  and  $D = N = 4$  SUSY algebra can satisfy such conditions [4–6]. Since these successful examples are explained in the references in detail, it is rather instructive here to begin with the Leibniz rule conditions for the twisted  $N = 2$   $D = 4$  SUSY algebra and then see how the situations are improved by the dimensional reduction to the twisted  $N = 4$   $D = 3$ .

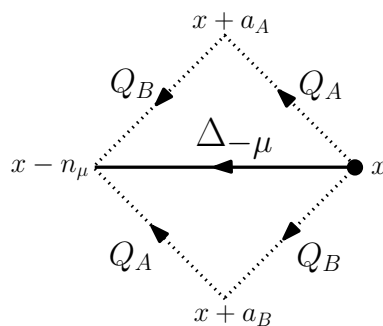
The twisted  $N = 2$   $D = 4$  SUSY algebra is given by

$$\{Q^+, Q_\mu^+\} = i\partial_\mu, \quad \{Q_{\rho\sigma}^+, Q_\mu^+\} = -i\delta_{\rho\sigma\mu\nu}^+\partial_\nu, \quad \{others\} = 0, \tag{4.8}$$

where  $\mu, \nu, \rho, \sigma$  run from 1 to 4. The symbols  $(Q^+, Q_\mu^+, Q_{\mu\nu}^+)$  denote the  $N = 2$   $D = 4$  twisted supercharges which transform as a scalar, a vector and a second rank self-dual



**Figure 1:** Lattice SUSY algebra subject to the condition (4.6)



**Figure 2:** Lattice SUSY algebra subject to the condition (4.7)

	$a_1^+$	$a_2^+$	$a_3^+$	$a_4^+$
$a^+$	$\pm n_1$	$\pm n_2$	$\pm n_3$	$\pm n_4$
$a_{12}^+$	$\pm n_2$	$\pm n_1$	$\pm n_4$	$\pm n_3$
$a_{13}^+$	$\pm n_3$	$\pm n_4$	$\pm n_1$	$\pm n_2$
$a_{14}^+$	$\pm n_4$	$\pm n_3$	$\pm n_2$	$\pm n_1$

**Table 3:** Leibniz rule conditions for the  $N = 2$   $D = 4$  twisted SUSY algebra

	$a_1^2$	$a_2^2$	$a_3^2$	$a^2$
$a^1$	$\pm n_1$	$\pm n_2$	$\pm n_3$	
$a_3^1$	$\pm n_2$	$\pm n_1$		$\pm n_3$
$a_2^1$	$\pm n_3$		$\pm n_1$	$\pm n_2$
$a_1^1$		$\pm n_3$	$\pm n_2$	$\pm n_1$

**Table 4:** Leibniz rule conditions after the dimensional reduction

tensor, respectively. The projector  $\delta_{\mu\nu\rho\sigma}^+ = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma}$  picks up only the self-dual part. The lattice Leibniz rule conditions associated with the lattice counterpart of (4.8) are expressed as, for example,

$$a^+ + a_1^+ = \pm n_1, \quad a_{12}^+ + a_1^+ = \pm n_2, \quad \text{etc.}, \quad (4.9)$$

where the signs are chosen to be positive (negative) if the corresponding difference operator is the forward (backward) type. All the conditions associated with the twisted  $N = 2$   $D = 4$  algebra are summarized in table 3. Viewing these conditions, one can easily notice that they are actually over-constrained. For example, the total sum of  $a_A$  along the diagonal part of table 3 gives,

$$\begin{aligned} \sum a_A &= (a^+ + a_1^+) + (a_{12}^+ + a_2^+) + (a_{13}^+ + a_3^+) + (a_{14}^+ + a_4^+) \\ &= \pm n_1 \pm n_1 \pm n_1 \pm n_1 \propto n_1 \text{ or } 0, \end{aligned} \quad (4.10)$$

while one of the off-diagonal combinations give rise to the different value of the total sum,

$$\begin{aligned} \sum a_A &= (a_{13}^+ + a_1^+) + (a_{12}^+ + a_2^+) + (a_{14}^+ + a_3^+) + (a^+ + a_4^+) \\ &= \pm n_3 \pm n_1 \pm n_2 \pm n_4, \end{aligned} \quad (4.11)$$

which indicates that the  $a_A$  cannot have any definite values. The lattice Leibniz rule conditions for twisted  $N = 2$   $D = 4$  thus do not have any consistent solutions.

Now we perform a dimensional reduction to three dimensions by truncating the fourth dimension. By employing the  $SU(2)_R \times SU(2)_N$  manifestly covariant notation in the last

section, the  $N = 4$   $D = 3$  twisted SUSY algebra of either the A-type or the B-type can be written as

$$\{Q^a, Q_\mu^b\} = +i\epsilon^{ab}\partial_\mu, \quad \{Q_\mu^a, Q_\nu^b\} = -i\epsilon^{ab}\epsilon_{\mu\nu\rho}\partial_\rho. \quad (4.12)$$

The Leibniz rule conditions for the lattice counterpart of (4.12) is summarized in table 4. Notice that the off-diagonal summations such as (4.11) turn to be irrelevant after the dimensional reduction since there are no conditions arising from the fourth direction. Taking a look at the total sum of  $a_A$ 's in terms of the relevant combinations, one could realize that the total sum of  $a_A$ 's should vanish in order for the  $N = 4$   $D = 3$  Leibniz rule conditions to be satisfied, which means that one should have two forward difference and two backward difference operators for each direction. One of the possible choices for the  $N = 4$   $D = 3$  lattice SUSY algebra is thus given by, for example,

$$\{Q^a, Q_\mu^b\} = +i\epsilon^{ab}\Delta_{+\mu}, \quad \{Q_\mu^a, Q_\nu^b\} = -i\epsilon^{ab}\epsilon_{\mu\nu\rho}\Delta_{-\rho}, \quad (4.13)$$

which is associated with the Leibniz rule condition,

$$a^a + a_\mu^b = +|\epsilon^{ab}|n_\mu, \quad a_\mu^a + a_\nu^b = -|\epsilon^{ab}|\epsilon_{\mu\nu\rho}n_\rho. \quad (4.14)$$

As is presented in [6], the consistent solutions for the conditions (4.14) are given by

$$a^1 = (\text{arbitrary}), \quad a_\mu^2 = +n_\mu - a^1, \quad (4.15)$$

$$a_\mu^1 = -\sum_{\lambda \neq \mu} n_\lambda + a^1, \quad a^2 = +\sum_{\lambda=1}^3 n_\lambda - a^1. \quad (4.16)$$

As is also stressed in [6], the eight supercharges of  $N = 4$   $D = 3$  twisted SUSY algebra have one-to-one correspondences with all the possible simplicial elements in three dimensions, namely, 0-form, 1-form, 2-form and 3-form whose total number of components is  $1 + 3 + 3 + 1 = 8$ . This geometrical consistency with the Dirac-Kähler picture of the fermions essentially provides the reason why the  $N = 4$   $D = 3$  twisted algebra can be exactly realized on the three dimensional lattice. From this viewpoint, there is no wonder why the  $N = 2$   $D = 4$  lattice Leibniz rule conditions do not have any consistent solutions, because in four dimensions we have 0-form, 1-form, 2-form, 3-form and 4-form whose total number of components is  $1 + 4 + 6 + 4 + 1 = 16$ . We obviously need the Dirac-Kähler twisted  $N = D = 4$  SUSY algebra with sixteen supercharges to be exactly realized on the four dimensional lattice, as is already pointed out in [5].

Let us now turn to address the possibility of formulating the A-type twisted SYM on the lattice. We first remind that, in the manifestly gauge covariant formulation of the lattice SYM [5, 6], each supercharge  $(Q_A)_{x+a_A,x}$  is replaced by the corresponding fermionic gauge link variable  $(\nabla_A)_{x+a_A,x}$  whose gauge variation is given by

$$(\nabla_A)_{x+a_A,x} \rightarrow G_{x+a_A}(\nabla_A)_{x+a_A,x}G_x^{-1}, \quad (4.17)$$

where  $G_x$  denotes the finite gauge transformation at the site  $x$ . In order for the SYM multiplet to be realized on the lattice, we need to take care of not only the Leibniz rule

itself but also the lattice gauge covariance of the entire SYM multiplet as well. Although there is no distinction between the A-type and the B-type in the lattice realizations of the SUSY algebra itself (4.13) except for interchanging the roles of the  $SU(2)_R$  and the  $SU(2)_N$ , the situation becomes quite different when one comes to the gauge covariance on the lattice.

For the B-type twisted SYM which is subject to the constraints (3.38) and (3.39), we can successfully introduce the bosonic gauge link variables,  $(\mathcal{U}_{\pm\mu})_{x\pm n_\mu, x} = \left( e^{\pm i(A_\mu \pm \phi^{(\mu)})} \right)_{x\pm n_\mu, x}$ , as the lattice realization of the gauge covariant derivatives,  $\mp(\partial_\mu - i(A_\mu \pm \phi^{(\mu)}))$  [6]. Note that the scalar fields  $\phi^{(\mu)}$ , transforming as a three dimensional vector after the twisting, are embedded in the bosonic gauge link variables  $\mathcal{U}_{\pm\mu}$ . In terms of these link variables, the lattice counterpart of the constraints (3.38) and (3.39) can be expressed as

$$\{\nabla^a, \nabla_\mu^b\}_{x+a^a+a_\mu^b, x} = +i\epsilon^{ab}(\mathcal{U}_{+\mu})_{x+n_\mu, x}, \tag{4.18}$$

$$\{\nabla_\mu^a, \nabla_\nu^b\}_{x+a_\mu^a+a_\nu^b, x} = +i\epsilon^{ab}\epsilon_{\mu\nu\rho}(\mathcal{U}_{-\rho})_{x-n_\rho, x}, \tag{4.19}$$

$$\{\text{others}\} = 0, \tag{4.20}$$

where the anti-commutators in the l.h.s. are defined as the link anti-commutators as in the supercharge case (4.4),

$$\{\nabla^a, \nabla_\mu^b\}_{x+a^a+a_\mu^b, x} \equiv (\nabla^a)_{x+a^a+a_\mu^b, x+a_\mu^b}(\nabla_\mu^b)_{x+a_\mu^b, x} + (\nabla_\mu^b)_{x+a_\mu^b+a^a, x+a^a}(\nabla^a)_{x+a^a, x} \tag{4.21}$$

Notice that the Leibniz rule conditions (4.14) are nothing but the gauge covariance conditions for the B-type twisted SYM constraints (4.18) and (4.19). Furthermore, once the starting constraints are given in a gauge covariant manner, all the analyses of the Jacobi identities automatically respect the gauge covariance on the lattice thanks to the link definition of the (anti-)commutators. In this way, the gauge covariance for the B-type twisted SYM is manifestly maintained on the lattice.<sup>4</sup>

As for the A-type twisted multiplet in contrast, one finds that the scalar fields  $\phi^{ab}$  are embedded in the constraints (3.20) and (3.21) as follows,

$$\{\nabla^a, \nabla^b\} = \{\nabla_1^a, \nabla_1^b\} = \{\nabla_2^a, \nabla_2^b\} = \{\nabla_3^a, \nabla_3^b\} = -i\phi^{ab}, \tag{4.22}$$

This “diagonal” embedding of the scalar fields is directly related the fact that the  $\phi^{ab}$  are transforming as scalars under  $(SO(3)_E \times SO(3)_R)_{\text{diag}}$  even after the twisting (See (3.30)). The crucial observation here is that the equalities in (4.22) can never be simultaneously satisfied on the lattice since the solutions for the lattice Leibniz rule (4.15)–(4.16) indicate that each anti-commutator should be located on a different link each other. For example, in the case of  $a \neq b$  we have the four anti-commutators obviously located on different links each other,

$$\{\nabla^a, \nabla^b\}_{x+n_1+n_2+n_3, x}, \{\nabla_1^a, \nabla_1^b\}_{x+n_1-n_2-n_3, x}, \{\nabla_2^a, \nabla_2^b\}_{x-n_1+n_2-n_3, x}, \{\nabla_3^a, \nabla_3^b\}_{x-n_1-n_2+n_3, x}. \tag{4.23}$$

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<sup>4</sup>As is claimed in [6], we need to introduce covariantly constant fermionic parameters  $\eta_A$  in order to maintain the gauge covariance associated with the twisted SUSY variations of the component fields on the lattice.

Likewise, we do not have any chance to simultaneously satisfy the equalities in (4.22) in the case of  $a = b$ , either. Thus, the scalar fields  $\phi^{ab}$  in the A-type twisted multiplet can never be located on any definite links on the three dimensional lattice. We conclude that, amongst two types of the  $N = 4 D = 3$  twisted SYM, only the B-type twisted multiplet can be realized on the lattice keeping the Leibniz rule and the gauge covariance on the lattice. Namely, the formulation given in [6] is the unique lattice realization of  $N = 4 D = 3$  twisted SYM satisfying these criteria.

One may wonder why the A-type multiplet fails on the lattice even though the starting constraints for the A-type (3.20)–(3.21) and the B-type (3.38)–(3.39) are related each other by the linear combinations (3.32)–(3.35). One should remind here that although the A-type and B-type basis are related each other, those two give rise to the inequivalent theories after the twisting since they respect the different diagonal subgroups of the Lorentz and the internal rotations. The above analysis suggests that only the B-type twisted rotational subgroup can be survived on the three dimensional lattice consistently with the gauge covariance of the lattice SUSY multiplet.

It should be mentioned here that we had a similar situation also in the twisted  $N = D = 2$ . The existence of two inequivalent twists in the  $N = D = 2$  SYM is originated from the two inequivalent internal symmetries, the ghost number  $\mathcal{U}(1)$  and the  $\text{SO}(2)_R$  internal rotation.<sup>5</sup> In the lattice formulation proposed in [5], we introduced the scalar fields in such a way that they transform as a two dimensional vector after the twisting. We could maintain the gauge covariance of the  $N = D = 2$  twisted SYM multiplet on the lattice by embedding the gauge fields and the scalar fields in the bosonic gauge link variables just as in the B-type twist of  $N = 4 D = 3$  explained above. The other twisted basis of  $N = D = 2$  turned out not to accommodate the lattice gauge covariance just as in the case of the A-type twisted  $N = 4 D = 3$ . The lattice formulation given in [5] is thus providing the unique lattice realization of  $N = D = 2$  twisted SYM compatible with the lattice Leibniz rule and the gauge covariance on the lattice.

## 5. Summary & discussions

An entirely off-shell formulation of the  $N = 4 D = 3$  twisted SYM is presented. We employ the twisted superconnection method in order to provide an manifestly gauge covariant off-shell framework. Although the formulation is given in terms of the twisted basis, the resulting SYM action respects the entire symmetry group of the three dimensional Lorentz rotations  $\text{SU}(2)_E$  and the internal rotations  $\text{SU}(2)_R$  in the flat spacetime. We also explore the two inequivalent twisted SYM of  $N = 4 D = 3$  rather explicitly and we then show that the recent proposal of  $N = 4 D = 3$  twisted SYM on the lattice [6] is essentially classified as the B-type twisted SYM. We also consider the possibility of realizing the  $N = 2 D = 4$  twisted SYM as well as the  $N = 4 D = 3$  A-type twisted SYM on the lattice by analyzing the lattice Leibniz rule and the gauge covariance on the lattice. We then show that the  $N = 2 D = 4$  twisted SUSY algebra cannot satisfy the Leibniz rule conditions on the

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<sup>5</sup>The author thanks I. Kanamori for his comments and discussions.



four dimensional lattice and that the A-type twisted SYM multiplet cannot be compatible with the gauge covariance on the three dimensional lattice. The analyses show that the lattice formulation given in [6] is the unique realization of  $N = 4$   $D = 3$  twisted SYM on the lattice satisfying these criteria. In the same respect, we also mentioned that the two dimensional lattice SYM formulation given in [4] is providing the unique formulation of  $N = D = 2$  twisted SYM on the lattice.

In this paper, we also explicitly derive the twisted SUSY exact relation between  $N = 4$   $D = 3$  twisted SYM and the super Chern-Simons entirely in the off-shell regime. Thanks to the off-shell structure, we clarify the twisted SUSY invariant nature of these super Chern-Simons actions. We point out that the existence of the sub-algebra and the sub-multiplet in the  $N = 4$   $D = 3$  is responsible for the twisted SUSY invariance. We observe that these relations are also playing important roles when studying a possible formulation of the super Chern-Simons on the lattice. The result of this analysis will be given elsewhere.

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## References

- [1] E. Witten, *Topological quantum field theory*, *Commun. Math. Phys.* **117** (1988) 353.
- [2] E. Witten, *Topological  $\sigma$ -models*, *Commun. Math. Phys.* **118** (1988) 411.
- [3] E. Witten, *Topological gravity*, *Phys. Lett.* **B 206** (1988) 601.
- [4] A. D’Adda, I. Kanamori, N. Kawamoto and K. Nagata, *Twisted superspace on a lattice*, *Nucl. Phys.* **B 707** (2005) 100 [[hep-lat/0406029](#)]; *Twisted  $N = 2$  exact SUSY on the lattice for BF and Wess-Zumino*, *Nucl. Phys.* **140** (Proc. Suppl.) (2005) 754 [[hep-lat/0409092](#)];  *$N = D = 2$  twisted supersymmetry on a lattice*, *Nucl. Phys.* **140** (Proc. Suppl.) (2005) 757.
- [5] A. D’Adda, I. Kanamori, N. Kawamoto and K. Nagata, *Exact extended supersymmetry on a lattice: twisted  $N = 2$  super Yang-Mills in two dimensions*, *Phys. Lett.* **B 633** (2006) 645 [[hep-lat/0507029](#)].
- [6] A. D’Adda, I. Kanamori, N. Kawamoto and K. Nagata, *Exact extended supersymmetry on a lattice: twisted  $N = 4$  super Yang-Mills in three dimensions*, [arXiv:0707.3533](#); *Lattice formulation of the  $N = 4$   $D = 3$  twisted super Yang-Mills*, *PoS(LATTICE 2007)*271 [[arXiv:0709.0722](#)].
- [7] S. Catterall and S. Karamov, *Exact lattice supersymmetry: the two-dimensional  $N = 2$  Wess-Zumino model*, *Phys. Rev.* **D 65** (2002) 094501 [[hep-lat/0108024](#)]; *A lattice study of the two-dimensional Wess Zumino model*, *Phys. Rev.* **D 68** (2003) 014503 [[hep-lat/0305002](#)]; S. Catterall, *Lattice supersymmetry and topological field theory*, *JHEP* **05** (2003) 038 [[hep-lat/0301028](#)];

- S. Catterall and S. Ghadab, *Lattice  $\sigma$ -models with exact supersymmetry*, *JHEP* **05** (2004) 044 [[hep-lat/0311042](#)]; *Twisted supersymmetric  $\sigma$ -model on the lattice*, *JHEP* **10** (2006) 063 [[hep-lat/0607010](#)];
- S. Catterall, *A geometrical approach to  $N = 2$  super Yang-Mills theory on the two dimensional lattice*, *JHEP* **11** (2004) 006 [[hep-lat/0410052](#)]; *Lattice formulation of  $N = 4$  super Yang-Mills theory*, *JHEP* **06** (2005) 027 [[hep-lat/0503036](#)]; *Simulations of  $N = 2$  super Yang-Mills theory in two dimensions*, *JHEP* **03** (2006) 032 [[hep-lat/0602004](#)]; *On the restoration of supersymmetry in twisted two-dimensional lattice Yang-Mills theory*, *JHEP* **04** (2007) 015 [[hep-lat/0612008](#)];
- S. Catterall and T. Wiseman, *Towards lattice simulation of the gauge theory duals to black holes and hot strings*, *JHEP* **12** (2007) 104 [[arXiv:0706.3518](#)]; *Gauge-gravity duality – super Yang-Mills quantum mechanics*, PoS(LATTICE 2007)051 [[arXiv:0709.3497](#)];
- M. Hanada, J. Nishimura and S. Takeuchi, *Non-lattice simulation for supersymmetric gauge theories in one dimension*, *Phys. Rev. Lett.* **99** (2007) 161602 [[arXiv:0706.1647](#)];
- K.N. Anagnostopoulos, M. Hanada, J. Nishimura and S. Takeuchi, *Monte Carlo studies of supersymmetric matrix quantum mechanics with sixteen supercharges at finite temperature*, [arXiv:0707.4454](#);
- F. Sugino, *A lattice formulation of super Yang-Mills theories with exact supersymmetry*, *JHEP* **01** (2004) 015 [[hep-lat/0311021](#)]; *Super Yang-Mills theories on the two-dimensional lattice with exact supersymmetry*, *JHEP* **03** (2004) 067 [[hep-lat/0401017](#)]; *Various super Yang-Mills theories with exact supersymmetry on the lattice*, *JHEP* **01** (2005) 016 [[hep-lat/0410035](#)]; *Two-dimensional compact  $N = (2, 2)$  lattice super Yang-Mills theory with exact supersymmetry*, *Phys. Lett. B* **635** (2006) 218 [[hep-lat/0601024](#)];
- M. Ünsal, *Twisted supersymmetric gauge theories and orbifold lattices*, *JHEP* **10** (2006) 089 [[hep-th/0603046](#)];
- K. Ohta and T. Takimi, *Lattice formulation of two dimensional topological field theory*, *Prog. Theor. Phys.* **117** (2007) 317 [[hep-lat/0611011](#)];
- T. Takimi, *Relationship between various supersymmetric lattice models*, *JHEP* **07** (2007) 010 [[arXiv:0705.3831](#)];
- P.H. Damgaard and S. Matsuura, *Relations among supersymmetric lattice gauge theories via orbifolding*, *JHEP* **08** (2007) 087 [[arXiv:0706.3007](#)];
- F. Bruckmann and M. de Kok, *Noncommutativity approach to supersymmetry on the lattice: SUSY quantum mechanics and an inconsistency*, *Phys. Rev. D* **73** (2006) 074511 [[hep-lat/0603003](#)];
- F. Bruckmann, S. Catterall and M. de Kok, *A critique of the link approach to exact lattice supersymmetry*, *Phys. Rev. D* **75** (2007) 045016 [[hep-lat/0611001](#)].
- [8] P. Dondi and H. Nicolai, *Lattice supersymmetry*, *Nuovo Cim.* **A41** (1977) 1;
- S. Elitzur, E. Rabinovici and A. Schwimmer, *Supersymmetric models on the lattice*, *Phys. Lett. B* **119** (1982) 165;
- T. Banks and P. Windey, *Supersymmetric lattice theories*, *Nucl. Phys. B* **198** (1982) 226;
- S. Cecotti and L. Girardello, *Stochastic processes in lattice (extended) supersymmetry*, *Nucl. Phys. B* **226** (1983) 417;
- N. Sakai and M. Sakamoto, *Lattice supersymmetry and the Nicolai mapping*, *Nucl. Phys. B* **229** (1983) 173;
- S. Elitzur and A. Schwimmer,  *$N = 2$  two-dimensional Wess-Zumino model on the lattice*, *Nucl. Phys. B* **226** (1983) 109;
- I. Ichinose, *Supersymmetric lattice gauge theory*, *Phys. Lett. B* **122** (1983) 68;
- J. Bartels and J.B. Bronzan, *Supersymmetry on a lattice*, *Phys. Rev. D* **28** (1983) 818;

- J. Bartels and G. Kramer, *A lattice version of the Wess-Zumino model*, *Z. Physik C* **20** (1983) 159;
- D.B. Kaplan, *Dynamical generation of supersymmetry*, *Phys. Lett.* **B 136** (1984) 162;
- R. Nakayama and Y. Okada, *Supercurrent anomaly in lattice gauge theory*, *Phys. Lett.* **B 134** (1984) 241;
- S. Nojiri, *Continuous 'translation' and supersymmetry on the lattice*, *Prog. Theor. Phys.* **74** (1985) 819;
- G. Curci and G. Veneziano, *Supersymmetry and the lattice: a reconciliation?*, *Nucl. Phys.* **B 292** (1987) 555;
- M. Golterman and D. Petcher, *A local interactive lattice model with supersymmetry*, *Nucl. Phys.* **B 319** (1989) 307.
- [9] J. Nishimura, *Four-dimensional  $N = 1$  supersymmetric Yang-Mills theory on the lattice without fine-tuning*, *Phys. Lett.* **B 406** (1997) 215 [[hep-lat/9701013](#)];
- N. Maru and J. Nishimura, *Lattice formulation of supersymmetric Yang-Mills theories without fine-tuning*, *Int. J. Mod. Phys.* **A 13** (1998) 2841 [[hep-th/9705152](#)];
- H. Neuberger, *Vector like gauge theories with almost massless fermions on the lattice*, *Phys. Rev.* **D 57** (1998) 5417 [[hep-lat/9710089](#)];
- D.B. Kaplan and M. Schmaltz, *Supersymmetric Yang-Mills theories from domain wall fermions*, *Chin. J. Phys.* **38** (2000) 543 [[hep-lat/0002030](#)];
- G.T. Fleming, J.B. Kogut and P.M. Vranas, *Super Yang-Mills on the lattice with domain wall fermions*, *Phys. Rev.* **D 64** (2001) 034510 [[hep-lat/0008009](#)];
- I. Montvay, *Supersymmetric Yang-Mills theory on the lattice*, *Int. J. Mod. Phys.* **A 17** (2002) 2377 [[hep-lat/0112007](#)].
- [10] D.B. Kaplan, E. Katz and M. Ünsal, *Supersymmetry on a spatial lattice*, *JHEP* **05** (2003) 037 [[hep-lat/0206019](#)];
- A.G. Cohen, D.B. Kaplan, E. Katz and M. Ünsal, *Supersymmetry on a euclidean spacetime lattice. I: a target theory with four supercharges*, *JHEP* **08** (2003) 024 [[hep-lat/0302017](#)];
- Supersymmetry on a euclidean spacetime lattice. II: target theories with eight supercharges*, *JHEP* **12** (2003) 031 [[hep-lat/0307012](#)];
- J. Nishimura, S.-J. Rey and F. Sugino, *Supersymmetry on the noncommutative lattice*, *JHEP* **02** (2003) 032 [[hep-lat/0301025](#)];
- J. Giedt, E. Poppitz and M. Rozali, *Deconstruction, lattice supersymmetry, anomalies and branes*, *JHEP* **03** (2003) 035 [[hep-th/0301048](#)];
- J. Giedt, *Non-positive fermion determinants in lattice supersymmetry*, *Nucl. Phys.* **B 668** (2003) 138 [[hep-lat/0304006](#)];
- The fermion determinant in  $(4, 4)$  2d lattice super-Yang-Mills*, *Nucl. Phys.* **B 674** (2003) 259 [[hep-lat/0307024](#)];
- Deconstruction and other approaches to supersymmetric lattice field theories*, *Int. J. Mod. Phys.* **A 21** (2006) 3039 [[hep-lat/0602007](#)];
- Quiver lattice supersymmetric matter, D1/D5 branes and  $AdS_3/CFT_2$* , [hep-lat/0605004](#);
- D.B. Kaplan and M. Ünsal, *A euclidean lattice construction of supersymmetric Yang-Mills theories with sixteen supercharges*, *JHEP* **09** (2005) 042 [[hep-lat/0503039](#)];
- M. Ünsal, *Compact gauge fields for supersymmetric lattices*, *JHEP* **11** (2005) 013 [[hep-lat/0504016](#)];
- Supersymmetric deformations of type IIB matrix model as matrix regularization of  $N = 4$  SYM*, *JHEP* **04** (2006) 002 [[hep-th/0510004](#)];
- T. Onogi and T. Takimi, *Perturbative study of the supersymmetric lattice theory from matrix model*, *Phys. Rev.* **D 72** (2005) 074504 [[hep-lat/0506014](#)];

- M.G. Endres and D.B. Kaplan, *Lattice formulation of (2, 2) supersymmetric gauge theories with matter fields*, *JHEP* **10** (2006) 076 [[hep-lat/0604012](#)];
- P.H. Damgaard and S. Matsuura, *Classification of supersymmetric lattice gauge theories by orbifolding*, *JHEP* **07** (2007) 051 [[arXiv:0704.2696](#)];
- S. Matsuura, *Exact vacuum energy of orbifold lattice theories*, *JHEP* **12** (2007) 048 [[arXiv:0709.4193](#)].
- [11] W. Bietenholz, *Exact supersymmetry on the lattice*, *Mod. Phys. Lett. A* **14** (1999) 51 [[hep-lat/9807010](#)];
- K. Fujikawa and M. Ishibashi, *Lattice chiral symmetry and the Wess-Zumino model*, *Nucl. Phys. B* **622** (2002) 115 [[hep-th/0109156](#)]; *Lattice chiral symmetry, Yukawa couplings and the Majorana condition*, *Phys. Lett. B* **528** (2002) 295 [[hep-lat/0112050](#)];
- Y. Kikukawa and Y. Nakayama, *Nicolai mapping vs. exact chiral symmetry on the lattice*, *Phys. Rev. D* **66** (2002) 094508 [[hep-lat/0207013](#)];
- K. Fujikawa,  *$N = 2$  Wess-Zumino model on the  $D = 2$  euclidean lattice*, *Phys. Rev. D* **66** (2002) 074510 [[hep-lat/0208015](#)]; *Supersymmetry on the lattice and the Leibniz rule*, *Nucl. Phys. B* **636** (2002) 80 [[hep-th/0205095](#)];
- M. Bonini and A. Feo, *Wess-Zumino model with exact supersymmetry on the lattice*, *JHEP* **09** (2004) 011 [[hep-lat/0402034](#)];
- J.W. Elliott and G.D. Moore, *Three dimensional  $N = 2$  supersymmetry on the lattice*, *PoS(LAT2005)245* [[hep-lat/0509032](#)];
- K. Itoh, M. Kato, H. Sawanaka, H. So and N. Ukita, *Novel approach to super Yang-Mills theory on lattice: exact fermionic symmetry and 'Ichimatsu' pattern*, *JHEP* **02** (2003) 033 [[hep-lat/0210049](#)]; *Towards the super Yang-Mills theory on the lattice*, *Prog. Theor. Phys.* **108** (2002) 363 [[hep-lat/0112052](#)];
- A. Feo, *Supersymmetry on the lattice*, *Nucl. Phys.* **119** (Proc. Suppl.) (2003) 198 [[hep-lat/0112052](#)]; *Supersymmetry on the lattice: where do predictions and results stand?*, [hep-lat/0311037](#); *Predictions and recent results in SUSY on the lattice*, *Mod. Phys. Lett. A* **19** (2004) 2387 [[hep-lat/0410012](#)];
- H. Suzuki and Y. Taniguchi, *Two-dimensional  $N = (2, 2)$  super Yang-Mills theory on the lattice via dimensional reduction*, *JHEP* **10** (2005) 082 [[hep-lat/0507019](#)];
- H. Suzuki, *Two-dimensional  $N = (2, 2)$  super Yang-Mills theory on computer*, *JHEP* **09** (2007) 052 [[arXiv:0706.1392](#)];
- H. Fukaya, I. Kanamori, H. Suzuki and T. Takimi, *Numerical results of two-dimensional  $N = (2, 2)$  super Yang-Mills theory*, *PoS(LATTICE 2007)264* [[arXiv:0709.4076](#)];
- Y. Kikukawa and H. Suzuki, *A local formulation of lattice Wess-Zumino model with exact  $U(1)_R$  symmetry*, *JHEP* **02** (2005) 012 [[hep-lat/0412042](#)];
- M. Harada and S. Pinsky,  *$N = 1$  super Yang-Mills on a  $(3 + 1)$  dimensional transverse lattice with one exact supersymmetry*, *Phys. Rev. D* **71** (2005) 065013 [[hep-lat/0411024](#)].
- [12] N. Kawamoto and T. Tsukioka,  *$N = 2$  supersymmetric model with Dirac-Kähler fermions from generalized gauge theory in two dimensions*, *Phys. Rev. D* **61** (2000) 105009 [[hep-th/9905222](#)].
- [13] J. Kato, N. Kawamoto and Y. Uchida, *Twisted superspace for  $N = D = 2$  super BF and Yang-Mills with Dirac-Kähler fermion mechanism*, *Int. J. Mod. Phys. A* **19** (2004) 2149 [[hep-th/0310242](#)].
- [14] J. Kato, N. Kawamoto and A. Miyake,  *$N = 4$  twisted superspace from Dirac-Kähler twist and off-shell SUSY invariant actions in four dimensions*, *Nucl. Phys. B* **721** (2005) 229 [[hep-th/0502119](#)].

- [15] S. Arianos, A. D'Adda, N. Kawamoto and J. Saito, *Lattice supersymmetry in 1D with two supercharges*, PoS(LATTICE 2007)259 [arXiv:0710.0487].
- [16] P.H. Damgaard and S. Matsuura, *Lattice supersymmetry: equivalence between the link approach and orbifolding*, *JHEP* **09** (2007) 097 [arXiv:0708.4129].
- [17] M. Blau and G. Thompson, *Aspects of  $N_T \geq 2$  topological gauge theories and d-branes*, *Nucl. Phys.* **B 492** (1997) 545 [hep-th/9612143].
- [18] B. Geyer and D. Mülsch,  *$N_T = 4$  equivariant extension of the 3d topological model of Blau and Thompson*, *Nucl. Phys.* **B 616** (2001) 476 [hep-th/0108042].
- [19] T. Eguchi and S.K. Yang,  *$N = 2$  superconformal models as topological field theories*, *Mod. Phys. Lett.* **A 5** (1990) 1693;  
J.M.F. Labastida and P.M. Llatas, *Topological matter in two-dimensions*, *Nucl. Phys.* **B 379** (1992) 220 [hep-th/9112051].
- [20] J.P. Yamron, *Topological actions from twisted supersymmetric theories*, *Phys. Lett.* **B 213** (1988) 325;  
N. Marcus, *The other topological twisting of  $N = 4$  Yang-Mills*, *Nucl. Phys.* **B 452** (1995) 331 [hep-th/9506002];  
C. Vafa and E. Witten, *A strong coupling test of  $S$  duality*, *Nucl. Phys.* **B 431** (1994) 3 [hep-th/9408074];  
J.M.F. Labastida and C. Lozano, *Mathai-Quillen formulation of twisted  $N = 4$  supersymmetric gauge theories in four dimensions*, *Nucl. Phys.* **B 502** (1997) 741 [hep-th/9702106].
- [21] J. Kato and A. Miyake, *Topological hypermultiplet on  $N = 2$  twisted superspace in four dimensions*, *Mod. Phys. Lett.* **A 21** (2006) 2569 [hep-th/0512269].
- [22] J. Wess and B. Zumino, *Superspace formulation of supergravity*, *Phys. Lett.* **B 66** (1977) 361;  
R. Grimm, M. Sohnius and J. Wess, *Extended supersymmetry and gauge theories*, *Nucl. Phys.* **B 133** (1978) 275;  
M.F. Sohnius, *Bianchi identities for supersymmetric gauge theories*, *Nucl. Phys.* **B 136** (1978) 461.
- [23] M. Alvarez and J.M.F. Labastida, *Topological matter in four-dimensions*, *Nucl. Phys.* **B 437** (1995) 356 [hep-th/9404115].
- [24] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton U.S.A. (1992).
- [25] P. Becher and H. Joos, *The Dirac-Kähler equation and fermions on the lattice*, *Z. Physik* **C 15** (1982) 343.
- [26] N. Seiberg and E. Witten, *Gauge dynamics and compactification to three dimensions*, hep-th/9607163.
- [27] I. Kanamori and N. Kawamoto, *Dirac-Kähler fermion from clifford product with noncommutative differential form on a lattice*, *Int. J. Mod. Phys.* **A 19** (2004) 695 [hep-th/0305094]; *Dirac-Kähler fermion with noncommutative differential forms on a lattice*, *Nucl. Phys.* **129** (Proc. Suppl.) (2004) 877 [hep-lat/0309120].